A MULTIPLICATIVE PROPERTY FOR ZERO-SUMS I

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Abstract. Let $G = C_n \times C_n$, where C_n denotes a cyclic group of order n, and let $k \in [0, n-1]$. We study the structure of sequences of terms from G with maximal length $|S| = 2n-2+k$ that fail to contain a nontrivial zero-sum subsequence of length at most $2n-1-k$. For $k \le 1$, this is the inverse question for the Davenport Constant. For $k = n-1$, this is the inverse question for the $\eta(G)$ invariant concerning short zero-sum subsequences. The structure in both these cases (known respectively as Property B and Property C) was established in a two-step process: first verifying the multiplicative property that, if the structural description holds when $n = n_1$ and $n = n_2$, then it holds when $n = n_1 n_2$, and then resolving the case n prime separately. When n is prime, the structural characterization for $k \in [2, \frac{2n+1}{3}]$ was recently established, showing S must have the form $S = e_1^{[n-1]} \cdot e_2^{[n-1]} \cdot (e_1+e_2)^{[k]}$ for some basis (e_1, e_2) for G. It was conjectured that this also holds for $k \in [2, n-2]$ (when n is prime). In this paper, we extend this conjecture by dropping the restriction that n be prime and establish the following multiplicative result. Suppose $k = k_m n + k_n$ with $k_m \in [0, m-1]$ and $k_n \in [0, n-1]$. If the conjectured structure holds for k_m in $C_m \times C_m$ and for k_n in $C_n \times C_n$, then it holds for k in $C_{mn} \times C_{mn}$. This reduces the full characterization question for n and k to the prime case. Combined with known results, this unconditionally establishes the structure for extremal sequences in $G = C_n \times C_n$ in many cases, including when n is only divisible by primes at most 7, when $n \ge 2$ is a prime power and $k \le \frac{2n+1}{3}$, or when n is composite and $k = n-d-1$ or $n-2d+1$ for a proper, nontrivial divisor $d \mid n$.

1. Introduction and Preliminaries

Regarding combinatorial notation for sequences and subsums, we utilize the standardized system surrounding multiplicative strings as outlined in the references [15] [14] [19]. For the reader new to this notational system, we begin with a self-contained review.

Notation. All intervals will be discrete, so for $x, y \in \mathbb{Z}$, we have $[x, y] = \{z \in \mathbb{Z} :\ x \le z \le y\}$. More generally, if G is an abelian group, $g \in G$, and $x, y \in \mathbb{Z}$, then

$$[x, y]_g = \{xg, (x+1)g, \ldots, yg\}.$$

For $G = C_n \oplus C_n$ a (ordered) **basis** for G is a pair (e_1, e_2) of elements $e_1, e_2 \in G$ such that $G = \langle e_1 \rangle \oplus \langle e_2 \rangle = C_n \oplus C_n$. For subsets $A_1, \ldots, A_k \subseteq G$, their sumset is defined as $A_1 + \ldots + A_k = \{a_1 + \ldots + a_k :\ a_i \in A_i \text{ for } i \in [1, k]\}$.

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Let G be an abelian group. In the tradition of Combinatorial Number Theory, a sequence of terms from G is a finite, unordered string of elements from G. We let $\mathcal{F}(G)$ denote the free abelian monoid with basis G, which consists of all (finite and unordered) sequences S of terms from G written as multiplicative strings using the boldsymbol $\boldsymbol{\cdot}$. This means a sequence $S \in \mathcal{F}(G)$ has the form

$$S = g_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_\ell$$

with $g_1, \ldots, g_\ell \in G$ the terms in S. Then

$$\mathsf{v}_g(S) = |\{i \in [1, \ell] : \ g_i = g\}|$$

denotes the multiplicity of the terms g in S, allowing us to represent a sequence S as

$$S = \prod^{\bullet}_{g \in G} g^{[\mathsf{v}_g(S)]},$$

where $g^{[n]} = \underbrace{g \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g}_{n}$ denotes a sequence consisting of the term $g \in G$ repeated $n \geq 0$ times. The maximum multiplicity of a term of S is the height of the sequence, denoted

$$\mathsf{h}(S) = \max\{\mathsf{v}_g(S) : \ g \in G\}.$$

The support of the sequence S is the subset of all elements of G that are contained in S, that is, that occur with positive multiplicity in S, which is denoted

$$\mathrm{Supp}(S) = \{g \in G : \ \mathsf{v}_g(S) > 0\}.$$

The length of the sequence S is

$$|S| = \ell = \sum_{g \in G} \mathsf{v}_g(S).$$

A sequence $T \in \mathcal{F}(G)$ with $\mathsf{v}_g(T) \leq \mathsf{v}_g(S)$ for all $g \in G$ is called a subsequence of S, denoted $T \mid S$, and in such case, $S \boldsymbol{\cdot} T^{[-1]} = T^{[-1]} \boldsymbol{\cdot} S$ denotes the subsequence of S obtained by removing the terms of T from S, so $\mathsf{v}_g(S \boldsymbol{\cdot} T^{[-1]}) = \mathsf{v}_g(S) - \mathsf{v}_g(T)$ for all $g \in G$.

Since the terms of S lie in an abelian group, we have the following notation regarding subsums of terms from S. We let

$$\sigma(S) = g_1 + \ldots + g_\ell = \sum_{g \in G} \mathsf{v}_g(S) g$$

denote the sum of the terms of S and call S a **zero-sum** sequence when $\sigma(S) = 0$. A **minimal zero-sum** sequence is a zero-sum sequence that cannot have its terms partitioned into two proper, nontrivial zero-sum subsequences. For $n \geq 0$, let

$$\Sigma_n(S) = \{\sigma(T) : \ T \mid S, \ |T| = n\}, \quad \Sigma_{\leq n}(S) = \{\sigma(T) : \ T \mid S, \ 1 \leq |T| \leq n\}, \quad \text{and}$$
$$\Sigma(S) = \{\sigma(T) : \ T \mid S, \ |T| \geq 1\}$$

denote the variously restricted collections of subsums of S. The sequence S is **zero-sum free** if $0 \notin \Sigma(S)$. Finally, if $\varphi : G \to G'$ is a map, then

$$\varphi(S) = \varphi(g_1) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} \varphi(g_\ell) \in \mathcal{F}(G')$$

denotes the sequence of terms from G' obtained by applying φ to each term from S.

Background. Let G be a finite abelian group. A classic topic in Combinatorial Number Theory is the study of conditions on sequences that ensure the existence of zero-sum subsequences with prescribed properties. Apart from the intrinsic combinatorial interest in such questions, they are also important when studying properties of factorization in Krull Domains and, more generally, in (Transfer) Krull Monoids. See [14] [15].

The most classic zero-sum invariant is the Davenport Constant $\mathsf{D}(G)$, defined as the minimal length such that any sequence of terms from G with length at least $\mathsf{D}(G)$ contains a nontrivial zero-sum subsequence. It is well-known that $\mathsf{D}(G)$ can be equivalently defined as the maximal length of a minimal zero-sum sequence. Indeed, $\mathsf{D}(G)-1$ is, by definition, the maximal length of a zero-sum free sequence T, and then one readily notes that $T\boldsymbol{\cdot}-\sigma(T)$ will be a minimal zero-sum sequence of length $\mathsf{D}(G)$. This shows there are minimal zero-sums of length $\mathsf{D}(G)$. Conversely, if S is any minimal zero-sum, then $S\boldsymbol{\cdot}g^{[-1]}$ is zero-sum free for any $g\in \mathrm{Supp}(G)$, ensuring no minimal zero-sum can have length exceeding $\mathsf{D}(G)$.

The precise value of $\mathsf{D}(G)$ is open in general and known only for a few small families of abelian groups, including p-groups and groups of rank at most two [14]. In particular [14, Theorem 5.8.3],

$$\mathsf{D}(C_n\oplus C_n)=2n-1$$

for $n\geq 1$. This is an old result of Olson [25] or van Emde Boas and Kruyswijk [6] whose proof required a more refined constant $\eta(G)$, defined as the minimal length such that any sequence of terms from G with length at least $\eta(G)$ contains a nontrivial zero-sum subsequence of length at most $\exp(G)$. For $G=C_n\oplus C_n$, we have [25] [6] [14, Theorem 5.8.3]

$$\eta(C_n\oplus C_n)=3n-2.$$

As a special case of a more general constant [3] [12], Delorme, Ordaz and Quiroz introduced [4] the refined constant $\mathsf{s}_{\leq\ell}(G)$ defined as the minimal length such that any sequence of terms from G with length at least $\mathsf{s}_{\leq\ell}(G)$ contains a nontrivial zero-sum subsequence of length at most ℓ, i.e.,

$$|S|\geq \mathsf{s}_{\leq\ell}(G)\quad\text{ implies }\quad 0\in\Sigma_{\leq\ell}(S).$$

Relations between $\mathsf{s}_{\leq\ell}(G)$ and Coding Theory may be found in [3], and other related works dealing with $\mathsf{s}_{\leq\ell}(G)$ include [7] [29] [11]. When $\ell<\exp(G)$, we have $\mathsf{s}_{\leq\ell}(G)=\infty$; when $\ell=\exp(G)$, we have $\mathsf{s}_{\leq\ell}(G)=\eta(G)$; and when $\ell\geq\mathsf{D}(G)$, we have $\mathsf{s}_{\leq\ell}(G)=\mathsf{D}(G)$. Thus, concerning the constant $\mathsf{s}_{\leq\ell}(G)$, the range of interest is $\ell\in[\exp(G),\mathsf{D}(G)]$, and $\mathsf{s}_{\leq\ell}(G)$ interpolates between the well-studied invariants $\eta(G)$ and $\mathsf{D}(G)$. For the case of $G=C_n\oplus C_n$, Chulin Wang and Kevin Zhao determined the exact value of $s_{\leq\ell}(G)$, showing [32]

$$\mathsf{s}_{\leq D-k}(C_n\oplus C_n)=D+k,\qquad\text{for } k\in[0,D-\exp(G)],$$

where $D = \mathsf{D}(C_n \oplus C_n)$. Since $\mathsf{D}(C_n \oplus C_n) = 2n - 1$, this can be restated as

$$\mathsf{s}_{\leq 2n-1-k}(C_n \oplus C_n) = 2n - 1 + k, \quad \text{for } k \in [0, n-1].$$

With the value $\mathsf{s}_{\leq 2n-1-k}(C_n \oplus C_n) = 2n - 1 + k$ established, there arises the associated inverse question characterizing all extremal sequences having maximal length $2n - 2 + k = \mathsf{s}_{\leq 2n-1-k}(C_n \oplus C_n) - 1$ with $0 \notin \Sigma_{\leq 2n-1-k}(S)$. For $k = 0$, this amounts to characterizing all zero-sum free sequences of maximal length $2n - 2 = \mathsf{D}(G) - 1$. For $k = 1$, this amounts to characterizing all minimal zero-sum sequences of maximal length $2n - 1 = \mathsf{D}(G)$. In view of our previous commentary, these two cases are equivalent to each other, the extremal sequences for $k = 0$ being simply the extremal sequences for $k = 1$ with any one term removed. For $k = n - 1$, this amounts to characterizing all extremal sequences of length $3n - 3 = \eta(G) - 1$ with $0 \notin \Sigma_{\leq n}(S)$.

The precise structure in both the case $k \leq 1$ and the case $k = n-1$ is known. For $k \leq 1$, this is achieved by combining the individual results of Gao, Geroldinger, Grynkiewicz and Reiher from [8] [10] [21] [28] with the numerical verification of the case when $n = 9$ [2]. The characterization of the extremal sequences for the Davenport constant has since proved quite useful, for instance being employed as machinery for the results in [1] [13] [16] [17] [26] [30] [27]. Since we will need to use both known cases heavily, we introduce some terminology.

A sequence S of terms from $G = C_n \oplus C_n$ is said to have **Property A** if there is a basis (e_1, e_2) for $G = C_n \oplus C_n$ such that $\text{Supp}(S) \subseteq \{e_1\} \cup \big(\langle e_1 \rangle + e_2\big)$. We say that the *group* $G = C_n \oplus C_n$ has **Property A** if every minimal zero-sum sequence S with $|S| = \mathsf{D}(G) = 2n - 1$ satisfies Property A. A sequence S of terms from $G = C_n \oplus C_n$ is said to have **Property B** if $\mathsf{h}(S) = \exp(G) - 1 = n - 1$, that is, S has some term e_1 with multiplicity $n - 1$. We say that the *group* $G = C_n \oplus C_n$ has **Property B** if every minimal zero-sum sequence S with $|S| = \mathsf{D}(G) = 2n - 1$ satisfies Property B. A simple argument shows that a minimal zero-sum sequence S with $|S| = \mathsf{D}(G) = 2n - 1$ satisfying Property A with basis (e_1, e_2) has the form

$$S = e_1^{n-1} \cdot {\prod}^\bullet_{i \in [1,n]} (x_i e_1 + e_2) \tag{1}$$

for some $x_1, \ldots, x_n \in [0, n-1]$ with $x_1 + \ldots + x_n \equiv 1 \mod n$. In particular, S satisfies Property B. It is also not hard to show (see [21]) that a minimal zero-sum sequence S with $|S| = \mathsf{D}(G) = 2n - 1$ that satisfies Property B, say with $\mathsf{v}_{e_1}(S) = \mathsf{h}(S) = n - 1$, has a basis (e_1, e_2) such that S has the form given in (1), and thus satisfies Property A with respect to the basis (e_1, e_2). Note, when S has two distinct elements e_1 and e_2 both with multiplicity $n - 1$, this ensures $S = e_1^{[n-1]} \cdot e_2^{[n-1]} \cdot (e_1 + e_2)$ with (e_1, e_2) and (e_2, e_1) both bases for G with respect to which S satisfies Property A.

Any sequence S having the form given in (1) is easily seen to be a minimal zero-sum sequence. The converse, that *every* minimal zero-sum sequence S of maximal length $|S| = \mathsf{D}(G) = 2n - 1$ must have the form given in (1), is the structural characterization of extremal sequences for the Davenport constant that was previously alluded to, which required several years and the combined effort of all the results from [8] [10] [21] [28] (as well as the individual verification of the case $n = 9$ [2]).

The precise structure of all extremal sequences for $k = n-1$ was achieved in [30] [9], and relies on the characterization in the case $k \leq 1$. We continue with the commonly used terminology in this case.

A sequence S of terms from $G = C_n \oplus C_n$ is said to have **Property C** if every term of S has multiplicity $n-1$. We say that the *group* $G = C_n \oplus C_n$ has **Property C** if every sequence S with $|S| = \eta(G) - 1 = 3n-3$ and $0 \notin \Sigma_{\leq n}(S)$ must satisfy Property C. It was shown in [9] that, assuming Property B (equivalently, property A) holds for G, then every sequence S with $|S| = \eta(G) - 1 = 3n-3$ and $0 \notin \Sigma_{\leq n}(S)$ must satisfy Property C, i.e., that Property A/B holding for $G = C_n \oplus C_n$ implies that Property C holds for G. Rather surprisingly, in contrast to the case for Property A/B, this does not easily yield a precise structural description of all possibilities for extremal sequences S when $k = n-1$. For $n = p$ prime, a derivation of the precise characterization from Property C can be found in [5], and the derivation of the precise characterization from Property C in the general case (when n may be composite) follows from a result of Schmid [30]. All such sequences satisfy Property A, and thus have the form

$$S = e_1^{[n-1]} \cdot e_2^{[n-1]} \cdot (xe_1 + e_2)^{[n-1]} \tag{2}$$

for some basis (e_1, e_2) for $G = C_n \oplus C_n$ and some $x \in [1, n-1]$ with $\gcd(x, n) = 1$.

In view of the discussion above, the inverse problem for $\mathsf{s}_{\leq 2n-1-k}(C_n \oplus C_n)$ is complete for the boundary values $k \leq 1$ and $k = n-1$. For the interior values $k \in [2, n-2]$ (and thus, for $n \geq 4$), a precise characterization of all extremal sequences S with length $|S| = \mathsf{s}_{\leq 2n-1-k}(C_n \oplus C_n) - 1$ but $0 \notin \Sigma_{\leq 2n-1-k}(S)$ is still open. There is partial progress in the case when $n = p$ is prime achieved in [23], where the precise structure is characterized for $G = C_p \oplus C_p$ when $k \in [2, \frac{2p+1}{3}]$ with $p \geq 5$, showing all such extremal sequences must have the form

$$S = e_1^{[p-1]} \cdot e_2^{[p-1]} \cdot (e_1 + e_2)^{[k]}$$

for some basis (e_1, e_2) for $G = C_p \oplus C_p$. It was conjectured in [23] [32] that the same structure should hold for any $k \in [2, p-2]$. Naturally extending this conjecture to composite values, we obtain the following conjecture that, if true, would fully characterize the structure of all extremal sequences for the zero-sum invariant $\mathsf{s}_{\leq 2n-1-k}(C_n \oplus C_n)$.

Conjecture 1.1. *Let $n \geq 2$, let $G = C_n \oplus C_n$, let $k \in [0, n-1]$, and let S be a sequence of terms from G with*

$$|S| = 2n - 2 + k \qquad \text{and} \qquad 0 \notin \Sigma_{\leq 2n-1-k}(S).$$

Then there exists a basis (e_1, e_2) for G such that the following hold.

1. *If $k = 0$, then $S \cdot g$ satisfies the description given in Item 2, where $g = -\sigma(S)$.*
2. *If $k = 1$, then*

$$S = e_1^{[n-1]} \cdot {\prod}^\bullet_{i \in [1, mn]} (x_i e_1 + e_2),$$

for some $x_1, \ldots, x_{mn} \in [0, n-1]$ *with* $x_1 + \ldots + x_{mn} \equiv 1 \mod n$.

3. *If* $k \in [2, n-2]$, *then*
$$S = e_1^{[n-1]} \cdot e_2^{[n-1]} \cdot (e_1 + e_2)^{[k]}.$$
4. *If* $k = n-1$, *then*
$$S = e_1^{[n-1]} \cdot e_2^{[n-1]} \cdot (xe_1 + e_2)^{[n-1]},$$
for some $x \in [1, n-1]$ *with* $\gcd(x, n) = 1$.

Per the discussion above, Parts 1, 2 and 4 in Conjecture 1.1 are known, and Part 3 holds when $n = p$ is prime and $k \leq \frac{2p+1}{3}$. It was also shown in [23] that Conjecture 1.1.3 holds when $n = p^s \geq 5$ is a prime power with $k \leq \frac{2n+1}{3}$ and $p \nmid k$. In general, we say that Conjecture 1.1 holds for k in $C_n \oplus C_n$ if Conjecture 1.1 is true when $G = C_n \oplus C_n$ for the given value $k \in [0, n-1]$. The main goal of this paper is Theorem 1.2, which shows that the structural description given in Conjecture 1.1.3 is multiplicative, thus reducing the full characterization problem for $\mathsf{s}_{\leq 2n-1-k}(C_n \oplus C_n)$ to the case when $n = p$ is prime, so the case when $G = C_p \oplus C_p$ with $p \geq 11$ prime (in view of Corollary 1.3). This reduction to the prime case is the main aim of the paper and emulates the strategy successfully used to characterize the extremal sequences for the Davenport Constant (the case $k \leq 1$), where the characterization problem was first reduced by a similar multiplicative result to the prime case [8] [10] [21], with the prime case later resolved by independent methods [28]. We remark that Schmid later reduced the characterization of extremal sequences for the Davenport Constant, in a general rank two abelian group, to the case $C_n \oplus C_n$ [31], and a forthcoming work [22] aims to similarly extend our methods to general rank two abelian groups.

Theorem 1.2. *Let* $n, m \geq 2$ *and let* $k \in [0, mn-1]$ *with* $k = k_m n + k_n$, *where* $k_m \in [0, m-1]$ *and* $k_n \in [0, n-1]$. *Suppose Conjecture 1.1 holds for* k_n *in* $C_n \oplus C_n$ *and either Conjecture 1.1 also holds for* k_m *in* $C_m \oplus C_m$ *or else* $k_n \geq 1$, $k_m \in [1, m-2]$ *and Conjecture 1.1 also holds for* $k_m + 1$ *in* $C_m \oplus C_m$. *Then Conjecture 1.1 holds for* k *in* $C_{mn} \oplus C_{mn}$.

While the reduction to the prime case is our main motivating goal, nonetheless, combining the known instances of Conjecture 1.1 with Theorem 1.2 yields many new cases where Conjecture 1.1 is established here without condition. In particular, we have the following corollaries, showing that Conjecture 1.1 is true when n is only divisible by primes at most 7, or when n is a prime power with $k \leq \frac{2n+1}{3}$, or when n is composite and $k = n - d - 1$ or $n - 2d + 1$ for a proper, nontrivial divisor $d \mid n$. The second corollary, in the case $m = 1$, removes the restriction $p \nmid k$ in [23, Theorem 5].

Corollary 1.3. *If* $n = 2^{s_1} 3^{s_2} 5^{s_3} 7^{s_4} \geq 2$ *with* $s_1, s_2, s_3, s_4 \geq 0$, *then Conjecture 1.1 holds in* $C_n \oplus C_n$ *for all* $k \in [0, n-1]$.

Corollary 1.4. *For any prime power* $n \geq 2$, *Conjecture 1.1 holds in* $C_n \oplus C_n$ *for all* $k \leq \frac{2n+1}{3}$.

Corollary 1.5. *For $n \geq 4$ composite with $d \mid n$ a proper, nontrivial divisor, Conjecture 1.1 holds for $k = n - d - 1$ and for $k = n - 2d + 1$ in $C_n \oplus C_n$.*

2. Preparatory Lemmas

The goal of this section is to collect together several properties about sequences having the structure given in Conjecture 1.1. However, we will also need the following two results. The first was a conjecture of Hamidoune established in [20, Theorem 1].

Theorem A. *Let G be a finite abelian group, let $k \geq 1$ and let $S \in \mathcal{F}(G)$ be a sequence with $|S| \geq |G| + 1$ and $k \leq |\operatorname{Supp}(S)|$. If $\mathsf{h}(S) \leq |G| - k + 2$ and $0 \notin \Sigma_{|G|}(S)$, then $|\Sigma_{|G|}(S)| \geq |S| - |G| + k - 1$.*

The second is [21, Lemma 3.2], which is the corrected version of [10, Proposition 4.2].

Theorem B. *Let $n \geq 2$, let $s \geq 3$ and let $G = C_n \oplus C_n$. If $S \in \mathcal{F}(G)$ is a zero-sum sequence with $|S| = sn - 1$ and $0 \notin \Sigma_{\leq n-1}(S)$, then there is a basis (e_1, e_2) for G such that either*

1. $\operatorname{Supp}(S) \subseteq \{e_1\} \cup \big(\langle e_1 \rangle + e_2\big)$ *and* $\mathsf{v}_{e_1}(S) \equiv -1 \mod n$, *or*
2. $S = e_1^{[an]} \boldsymbol{\cdot} e_2^{[bn-1]} \boldsymbol{\cdot} (xe_1 + e_2)^{[cn-1]} \boldsymbol{\cdot} (xe_1 + 2e_2)$ *for some $x \in [2, n-2]$ with $\gcd(x, n) = 1$, and some $a, b, c \geq 1$ with $a + b + c = s$.*

We begin now with a stability result for sequences satisfying Conjecture 1.1 when $k \geq 1$.

Lemma 2.1. *Let $n \geq 2$, let $k \in [1, n-1]$, let $G = C_n \oplus C_n$, and let $S \in \mathcal{F}(G)$ with $|S| = 2n-2+k$ and $0 \notin \Sigma_{\leq 2n-1-k}(S)$ such that Conjecture 1.1 holds for S. If $x \in \operatorname{Supp}(S)$, $y \in G$, and $S' = S \boldsymbol{\cdot} x^{[-1]} \boldsymbol{\cdot} y$ also has $0 \notin \Sigma_{\leq 2n-1-k}(S')$ with Conjecture 1.1 holding for S', then $x = y$.*

Proof. If $k = 1$, then S and S' satisfying the conclusion of Conjecture 1.1 implies they are both zero-sum sequences, which forces $x = y$. If $n = 2$, then $k = 1 \in [1, n-1]$ is forced. If $k \in [2, n-2]$, then $n \geq 4$ and $S = e_1^{[n-1]} \boldsymbol{\cdot} e_2^{[n-1]} \boldsymbol{\cdot} (e_1 + e_2)^{[k]}$ with $e_1 + (e_1 + e_2) \neq e_2$ and $e_2 + (e_1 + e_2) \neq e_1$ in view of $n \geq 3$. Since $n \geq 3$ and $k \geq 2$, we also guaranteed $e_1, e_2, e_1 + e_2 \in \operatorname{Supp}(S \boldsymbol{\cdot} x^{[-1]})$. Consequently, since S' also satisfies the conclusion of Conjecture 1.1, it must do so with respect to the basis (e_1, e_2), forcing $x = y$. Finally, if $k = n-1$ and $n \geq 3$, then $S = e_1^{[n-1]} \boldsymbol{\cdot} e_2^{[n-1]} \boldsymbol{\cdot} e_3^{[n-1]}$ and $\operatorname{Supp}(S) = \operatorname{Supp}(S \boldsymbol{\cdot} x^{[-1]}) \subseteq \operatorname{Supp}(S')$ in view of $n \geq 3$. Thus, since S' also satisfies the conclusion of Conjecture 1.1, it must do so with $\operatorname{Supp}(S') = \operatorname{Supp}(S)$, forcing $x = y$. □

We continue by showing how Property A implies the more detailed structure given in Conjecture 1.1.

Lemma 2.2. *Let $n \geq 4$, let $k \in [2, n-2]$, let $G = C_n \oplus C_n$, and let $S \in \mathcal{F}(G)$ be a sequence with $|S| = 2n - 2 + k$ and $0 \notin \Sigma_{\leq 2n-1-k}(S)$. Suppose there are $e_1, e_2 \in G$ with $\operatorname{Supp}(S) \subseteq \{e_1\} \cup \big(\langle e_1 \rangle + e_2\big)$. Then there is some $f_2 \in \langle e_1 \rangle + e_2$ such that (e_1, f_2) is a basis for G and $S = e_1^{[n-1]} \boldsymbol{\cdot} f_2^{[n-1]} \boldsymbol{\cdot} (e_1 + f_2)^{[k]}$.*

Proof. By hypothesis, $\mathrm{Supp}(S) \subseteq \{e_1\} \cup \big(\langle e_1\rangle + e_2\big) \subseteq \langle e_1, e_2\rangle$. Let $G' = \langle e_1, e_2\rangle \cong C_{m_1} \oplus C_{m_2}$ with $m_1 \mid m_2$. Let $k' = k + 2n - m_1 - m_2 \geq k$. If $G' = \langle e_1, e_2\rangle$ were a proper subgroup, then the hypotheses $|S| = 2n-2+k = m_1+m_2-1+(k'-1)$ ensures that S contains a nontrivial zero-sum with length at most $\max\{m_1+m_2-1-(k'-1), m_2\} = \max\{2(m_1+m_2-n)-k, m_2\} \leq \max\{2n-1-k, n\} = 2n-1-k$, contradicting the hypothesis $0 \notin \Sigma_{\leq 2n-1-k}(S)$. Therefore $G' = \langle e_1, e_2\rangle = G$, implying that (e_1, e_2) is a basis for G.

In view of our hypotheses, we have $S = e_1^{[\ell]} \boldsymbol{\cdot} \prod^{\bullet}_{i\in[1,2n-2+k-\ell]}(x_ie_1 + e_2)$ for some $\ell \geq 0$ and $x_i \in [0, n-1]$. We must have $\ell \leq n-1$, else S will contain an n-term zero-sum, contrary to the hypothesis $0 \notin \Sigma_{\leq 2n-1-k}(S)$. Let $S_2 = \prod^{\bullet}_{i\in[1,2n-2+k-\ell]} x_ie_1$ and $S_1 = e_1^{[\ell]}$. Then $|S_2| = 2n-2+k-\ell \geq n-1+k \geq n+1$. We also have $\mathsf{h}(S_2) \leq n-1$, else S again contains an n-term zero-sum, contrary to hypothesis. Thus $|\mathrm{Supp}(S_2)| \geq 2$.

Suppose $|S_1| = \ell \leq n-1-k$. Then the hypothesis $0 \notin \Sigma_{\leq 2n-1-k}(S)$ implies $0 \notin \Sigma_n(S_2) + (\Sigma(S_1) \cup \{0\})$, whence $\Sigma_n(S_2) \subseteq [1, n-1-\ell]_{e_1}$. In particular, $|\Sigma_n(S_2)| \leq n-1-\ell$. However, applying Theorem A to S_2 (using $k=2$), we obtain $|\Sigma_n(S_2)| \geq |S_2| - n + 1 = n-1+k-\ell > n-1-\ell$, contradicting what was just noted. So we can now assume $|S_1| = \ell \geq n-k$.

Since $|S_1| = \ell \geq n-k$, the hypothesis $0 \notin \Sigma_{\leq 2n-1-k}(S)$ implies that $0 \notin \Sigma_n(S_2) + (\Sigma_{\leq n-k-1}(S_1) \cup \{0\})$, whence $\Sigma_n(S_2) \subseteq [1,k]_{e_1}$. In particular, $|\Sigma_n(S_2)| \leq k$. Applying Theorem A to S_2 (using $k=2$), and then using the estimate $\ell \leq n-1$, we obtain $|\Sigma_n(S_2)| \geq |S_2| - n + 1 = n-1+k-\ell \geq k$. Thus equality must hold in all these estimates. In particular, $\Sigma_n(S_2) = [1,k]_{e_1}$, $\ell = n-1$, and $|\Sigma_n(S_2)| = |S_2| - n + 1$. It now follows from Theorem A applied to S_2 (using $k=3$) that $|\mathrm{Supp}(S_2)| = 2$.

Let $ye_1 \in \mathrm{Supp}(S_2)$ be an element with maximum multiplicity in S_2, and let $f_2 = ye_1 + e_2$. Then (e_1, f_2) is also a basis for G and

$$S = e_1^{[n-1]} \boldsymbol{\cdot} f_2^{[n-1-r]} \boldsymbol{\cdot} (xe_1 + f_2)^{k+r} \tag{3}$$

for some $x \in [1, n-1]$ and $r \in [0, \frac{n-1-k}{2}]$. Let $S_2' = 0^{[n-1-r]} \boldsymbol{\cdot} (xe_1)^{[k+r]}$. Repeating the argument of the previous paragraph using S_2' in place of S_2, we again conclude that $\Sigma_n(S_2') = [1,k]_{e_1}$. However, in view of the structure of S given by (3), we have $\Sigma_n(S_2') = (r+1)xe_1 + [0, k-1]_{xe_1}$. Thus

$$[1,k]_{e_1} = (r+1)xe_1 + [0,k-1]_{xe_1}. \tag{4}$$

Since $k \geq 2$, the set $[1,k]_{e_1}$ is not contained in a coset of a proper subgroup of $\langle e_1\rangle$. Hence (4) ensures $\langle xe_1\rangle = \langle e_1\rangle$. The left-hand side of (4) is an arithmetic progression with difference e_1 and length k, with $2 \leq k \leq n-2 = \mathrm{ord}(e_1) - 2$. It is well known and easily derived that, for such sets, the difference e_1 is unique up to sign. The right-hand side of (4) is also an arithmetic progression with difference xe_1 and length k, with $2 \leq k \leq n-2 = \mathrm{ord}(xe_1) - 2$. Thus, by the uniqueness of the difference, it follows that $xe_1 = \pm e_1$.

If $xe_1 = e_1$, then (4) forces $r = 0$ in view of $k < n$, yielding the desired structure for S. If $xe_1 = -e_1$, then (4) forces $r = n - k - 1$ in view of $k < n$. However, since $r \in [0, \frac{n-1-k}{2}]$, this in only possible if $k \geq n - 1$, which is contrary to hypothesis. □

The following lemma shows that the extension of a sequence satisfying Conjecture 1.1, obtained by concatenating an additional term, also satisfies Conjecture 1.1.

Lemma 2.3. *Let $n \geq 2$, let $k \in [1, n-1]$ with either $k = 1$ or $k \in [1, n-2]$, let $G = C_n \oplus C_n$, and let $S \in \mathcal{F}(G)$ be a sequence with $|S| = 2n-2+k$ and $0 \notin \Sigma_{\leq 2n-1-k}(S)$ such that Conjecture 1.1 holds for S. Suppose there is some $g \in G$ such that $0 \notin \Sigma_{\leq 2n-2-k}(S \cdot g)$. Then there exists a basis (e_1, e_2) for G such that $S \cdot g = e_1^{[n-1]} \cdot e_2^{[n-1]} \cdot (e_1 + e_2)^{[k+1]}$ with $g = e_1 + e_2$. In particular, Conjecture 1.1 holds for $S \cdot g$ (for $k \leq n-2$).*

Proof. Let (e_1, e_2) be an arbitrary basis for which Conjecture 1.1 holds for S. Let $g = x_1e_1 + x_2e_2$ with x_1, $x_2 \in [0, n-1]$.

Case 1: $k \in [2, n-2]$.

In this case, $n \geq 4$ and $S = e_1^{[n-1]} \cdot e_2^{[n-1]} \cdot (e_1 + e_2)^{[k]}$. By symmetry, we can w.l.o.g. assume $x_1 \geq x_2$. If $x_2 = 0$, then $S \cdot g$ contains n terms from $\langle e_1 \rangle \cong C_n$, and thus contains a zero-sum subsequence of length at most $\mathsf{D}(C_n) = n$, contradicting that $0 \notin \Sigma_{2n-2-k}(S \cdot g)$ (in view of $k \leq n-2$). Therefore $x_1 \geq x_2 \geq 1$. If $x_1 = x_2 = 1$, the desired conclusion follows, so we can assume $x_1 \geq 2$. If $x_1 \geq n-k$, then $e_2^{[x_1-x_2]} \cdot (e_1 + e_2)^{[n-x_1]} \cdot (x_1e_1 + x_2e_2)$ is a zero-sum subsequence of $S \cdot g$ with length $n - x_2 + 1 \leq n$, contradicting that $0 \notin \Sigma_{2n-2-k}(S \cdot g)$. On the other hand, if $x_2 \leq x_1 \leq n-k$, then $e_1^{[n-k-x_1]} \cdot e_2^{[n-k-x_2]} \cdot (e_1 + e_2)^{[k]} \cdot (x_1e_1 + x_2e_2)$ is a zero-sum subsequence of $S \cdot g$ with length $2n - k - x_1 - x_2 + 1 \leq 2n - 2 - k$ (with the latter inequality in view of $x_1 \geq 2$ and $x_2 \geq 1$), again contradicting that $0 \notin \Sigma_{2n-2-k}(S \cdot g)$.

Case 2: $k = 1$.

In this case, $n \geq 2$ and

$$S = e_1^{[n-1]} \cdot \prod\nolimits^{\bullet}_{i \in [1,n]} (y_ie_1 + e_2)$$

for some $y_1, \ldots, y_n \in [0, n-1]$ with $y_1 + \ldots + y_n \equiv 1 \mod n$. If $n = 2$, then $S = e_1 \cdot e_2 \cdot (e_1 + e_2)$, and our hypothesis $0 \notin \Sigma_{\leq 2n-2-k}(S \cdot g) = \Sigma_{\leq 1}(S \cdot g)$ simply means $g \neq 0$. In this case, replacing the basis (e_1, e_2) by a basis (f_1, f_2) with $g \notin \{f_1, f_2\}$, we find $S = f_1 \cdot f_2 \cdot (f_1 + f_2)$ with $g = f_1 + f_2$, and the desired result follows. Therefore we now assume $n \geq 3$, so $k = 1 \leq n - 2$.

If $x_2 = 0$, then $S \cdot g$ contains n terms from $\langle e_1 \rangle \cong C_n$, and thus contains a zero-sum subsequence of length at most $\mathsf{D}(C_n) = n$, contradicting that $0 \notin \Sigma_{2n-2-k}(S \cdot g)$ (in view of $k \leq n-2$). Therefore $x_2 \geq 1$.

Let $S_2 = \prod^{\bullet}_{i \in [1,n]} y_ie_1$. For any $(-x_1 + z)e_1 \in \Sigma_{n-x_2}(S_2)$, where $z \in [1, n]$, we have a subset $I \subseteq [1, n]$ with $|I| = n - x_2$ and $\sum\limits_{i \in I}(y_ie_1 + e_2) = (-x_1 + z)e_1 + (n - x_2)e_2$, meaning

$e_1^{[n-z]}\cdot(x_1e_1+x_2e_2)\cdot\prod^\bullet_{i\in I}(y_ie_1+e_2)$ is a zero-sum subsequence of $S\cdot g$ of length $2n-z+1-x_2$. Since $0\notin\Sigma_{\leq 2n-2-k}(S\cdot g)=\Sigma_{2n-3}(S\cdot g)$, this forces

$$z+x_2\leq 3. \tag{5}$$

Now S_2 is a sequence of n terms from a cyclic group of order n with $n-x_2\in[1,n-1]$. Moreover, since $y_1+\ldots+y_n\equiv 1\mod n$, we have $|\operatorname{Supp}(S_2)|\geq 2$.

If $T\mid S_2$ is any subsequence of length $n-x_2$, then $n-x_2\in[1,n-1]=[1,|S_2|-1]$ ensures that both T and $T^{[-1]}\cdot S_2$ contain at least one term, and since $|\operatorname{Supp}(S_2)|\geq 2$, it is thus possible to find terms $g\in\operatorname{Supp}(T)$ and $h\in\operatorname{Supp}(T^{[-1]}\cdot S_2)$ with $g\neq h$. This ensures that $T\cdot g^{[-1]}\cdot h$ is also a subsequence of S_2 with length $n-x_2$, and one with sum $\sigma(T)-g+h\neq\sigma(T)$. Thus $|\Sigma_{n-x_2}(S_2)|\geq 2$, meaning it is possible to find I as defined above with $z\geq 2$. Combined with (5) and $x_2\geq 1$, it follows that only $x_2=1$ is possible, whence $g=x_1e_1+e_2$. Since (e_1,xe_1+e_2) is also a basis for which Conjecture 1.1 holds for S, for any $x\in\mathbb{Z}$, we can replace the arbitrary basis (e_1,e_2) for which Conjecture 1.1 holds for S with the basis $(e_1,(x_1-1)e_1+e_2)$, thereby allowing us to w.l.o.g. assume $x_1=1$ in view of $e_1+\Big((x_1-1)e_1+e_2\Big)=g$. Thus we now have $g=e_1+e_2$ with $x_1=x_2=1$.

Since $x_2=1$, we have

$$\Sigma_{n-x_2}(S_2)=\Sigma_{n-1}(S_2)=\sigma(S_2)-\Sigma_1(S_2)=e_1-\operatorname{Supp}(S_2),$$

with the final inequality above in view of $y_1+\ldots+y_n\equiv 1\mod n$. Since $x_2=1$, (5) ensures that $z\leq 2$, which combined with $z\in[1,n]$ forces $z\in\{1,2\}$. Thus

$$e_1-\operatorname{Supp}(S_2)=\Sigma_{n-x_2}(S_2)\subseteq\{-x_1e_1+e_1,-x_1e_1+2e_1\}=\{0,e_1\},$$

whence

$$\operatorname{Supp}(S_2)=\{0,e_1\}$$

in view of $|\operatorname{Supp}(S_2)|\geq 2$. It follows that $y_i\equiv 0$ or $1\mod n$ for every $i\in[1,n]$. Letting $a\in[1,n-1]$ be the number of $i\in[1,n]$ with $y_i\equiv 1\mod m$, we find $1\equiv y_1+\ldots+y_n\equiv a+(n-a)(0)\mod n$, implying $a\equiv 1\mod n$. Thus $S=e_1^{[n-1]}\cdot e_2^{[n-1]}\cdot(e_1+e_2)$ with $g=e_1+e_2$, as desired. $\square$

The following lemma is the reverse of Lemma 2.3, showing that, if Conjecture 1.1 holds for a sequence and we remove a term, then Conjecture 1.1 also holds for the resulting subsequence.

Lemma 2.4. *Let $n\geq 3$, let $k\in[1,n-2]$, let $G=C_n\oplus C_n$, and let $S\in\mathcal{F}(G)$ be a sequence with $|S|=2n-2+k$ and $0\notin\Sigma_{\leq 2n-1-k}(S)$. Suppose there is some $g\in G$ such that $0\notin\Sigma_{\leq 2n-2-k}(S\cdot g)$ with Conjecture 1.1 holding for $S\cdot g$. Then there exists a basis (e_1,e_2) for G such that $S\cdot g=e_1^{[n-1]}\cdot e_2^{[n-1]}\cdot(e_1+e_2)^{[k+1]}$ with $g=e_1+e_2$. In particular, Conjecture 1.1 holds for S.*

Proof. Let (e_1,e_2) be an arbitrary basis for which Conjecture 1.1 holds for $S\cdot g$. Then

$$S\cdot g=e_1^{[n-1]}\cdot e_2^{[n-1]}\cdot(xe_1+e_2)^{[k+1]}$$

for some $x \in [1, n-1]$ with $\gcd(x, n) = 1$ and either $x = 1$ or $k = n-2$.

Suppose $x = 1$. In such case, if $g = e_1 + e_2$, the proof is complete, so either $g = e_2$ or $g = e_1$. But now $(e_1+e_2)^{[k+1]} \cdot e_1^{[n-k-1]} \cdot e_2^{[n-k-1]}$ is a zero-sum subsequence of S (in view of the hypothesis $k \geq 1$) with length $2n-1-k$, contradicting that $0 \notin \Sigma_{\leq 2n-1-k}(S)$. So we can now assume $x \geq 2$ with $k = n-2$, in which case

$$S \cdot g = e_1^{[n-1]} \cdot e_2^{[n-1]} \cdot (xe_1 + e_2)^{[n-1]}.$$

If $x = n-1$, then using the basis $(e_1, -e_1 + e_2)$ in place of (e_1, e_2), we find ourselves in the already completed case when $x = 1$. Thus we can assume $x \in [2, n-2]$ with $\gcd(x, n) = 1$, implying $n \geq 5$. Thus $k = n-2 \geq 3$ with $\mathrm{Supp}(S) \subseteq \{e_1\} \cup (\langle e_1 \rangle + e_2)$, $|S| = 2n-2+k$ and $0 \notin \Sigma_{\leq 2n-1-k}(S)$, allowing us to apply Lemma 2.2 to S to conclude that there is a basis (f_1, f_2) for G such that $S = f_1^{[n-1]} \cdot f_2^{[n-1]} \cdot (f_1+f_2)^{[k]}$. Since every term of $S \cdot g$ has multiplicity $n-1$, it follows that $g = f_1 + f_2$, and the desired conclusion follows. □

3. The Main Proof

We divide the proof of Theorem 1.2 into two main cases depending on the value of $k_n \in [0, n-1]$. We begin first with the case when $k_n \in [0, 1]$.

Proposition 3.1. *Let m, $n \geq 2$ and let $k \in [0, mn-1]$ with $k = k_m n + k_n$, where $k_m \in [0, m-1]$ and $k_n \in [0, 1]$. Suppose either Conjecture 1.1 holds for k_m in $C_m \oplus C_m$, or else $k_n = 1$, $k_m \in [1, m-2]$ and Conjecture 1.1 holds for $k_m + 1$ in $C_m \oplus C_m$. Then Conjecture 1.1 holds for k in $C_{mn} \oplus C_{mn}$.*

Proof. As remarked in the introduction, Conjecture 1.1 holds for $k \leq 1$ or $k = mn-1$ in every group $C_{mn} \oplus C_{mn}$. Therefore we can assume $k_m \in [1, m-1]$ and $k = k_m n + k_n \in [2, mn-2]$. Let $G = C_{mn} \oplus C_{mn}$ and let $S \in \mathcal{F}(G)$ be a sequence with

$$|S| = 2nm - 2 + k \quad \text{and} \quad 0 \notin \Sigma_{\leq 2nm-1-k}(S). \tag{6}$$

We need to show Conjecture 1.1.3 holds for S. Let $\varphi : G \to G$ be the multiplication by m homomorphism, so $\varphi(x) = mx$. Note

$$\varphi(G) = mG \cong C_n \oplus C_n \quad \text{and} \quad \ker \varphi = nG \cong C_m \oplus C_m.$$

If $k_n = 1$, set $S^* = S$. If $k_n = 0$, we can choose any element $g_0 \in -\sigma(S) + \ker \varphi$ and set $S^* = S \cdot g_0$. When $k_n = 0$, the definition of g_0 ensures that $\varphi(S^*)$ is zero-sum. When $k_n = 1$, we will shortly see below in Claim A that $\varphi(S^*)$ is also zero-sum. Note, in all cases,

$$|S^*| = 2mn - 1 + k_m n.$$

Define a *block decomposition* of S^* to be a factorization

$$S^* = W_0 \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$$

with $1 \le |W_i| \le n$ and $\varphi(W_i)$ zero-sum for each $i \in [1, 2m-2+k_m]$. Since $\mathsf{s}_{\le n}(\varphi(G)) = \mathsf{s}_{\le n}(C_n \oplus C_n) = 3n-2$ and $|S| \ge (2m-3+k_m)n + 3n - 2$, it follows by repeated application of the definition of $\mathsf{s}_{\le n}(\varphi(G))$ that S^* has a block decomposition, and one with $g_0 \in \mathrm{Supp}(W_0)$ when $k_n = 0$.

Claim A. If $S^* = W_0 \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ is a block decomposition, then $|W_i| = n$ for all $i \in [1, 2m-2+k_m]$, $\varphi(W_0)$ is a minimal zero-sum sequence of length $|W_0| = 2n-1$, and $0 \notin \Sigma_{\le n-1}(\varphi(S^*))$.

Proof. Suppose $k_n = 1$, so $S^* = S$. Let us show that $0 \notin \Sigma_{\le 2n-2}(\varphi(W_0))$. Assuming this fails, there is a nontrivial subsequence $W_0' \mid W_0$ with $|W_0'| \le 2n-2$ and $\varphi(W_0')$ zero-sum. Set $W_i' = W_i$ for $i \in [1, 2m-2+k_m]$. Then $S_\sigma = \sigma(W_0') \cdot \sigma(W_1') \cdot \ldots \cdot \sigma(W'_{2m-2+k_m})$ is a sequence of terms from $\ker \varphi \cong C_m \oplus C_m$ with $|S_\sigma| = 2m-1+k_m$. Since $\mathsf{s}_{\le 2m-1-k_m}(C_m \oplus C_m) = 2m-1+k_m$, it follows that S_σ has a nontrivial zero-sum subsequence of length at most $2m-1-k_m$, say $\prod^\bullet_{i\in I} \sigma(W_i')$ for some nonempty subset $I \subseteq [0, 2m-2+k_m]$ with $|I| \le 2m-1-k_m$. But now $\prod^\bullet_{i\in I} W_i'$ is a nontrivial zero-sum subsequence of $S^* = S$ with length

$$\sum_{i\in I} |W_i'| \le \max\{|W_0'|,\, n\} + (|I|-1)n \le 2n-2 + (2m-2-k_m)n = 2mn-1-k,$$

contradicting (6). This show that $0 \notin \Sigma_{2n-2}(\varphi(W_0))$. As a result,

$$|W_0| \le \mathsf{s}_{\le 2n-2}(\varphi(G)) - 1 = \mathsf{s}_{\le 2n-2}(C_n \oplus C_n) - 1 = 2n-1.$$

Suppose $k_n = 0$. Then $S^* = S \cdot g_0$ with $\varphi(S \cdot g_0)$ zero-sum by definition of g_0. Hence $\varphi(W_0)$ is also a zero-sum sequence. Let us show that $\varphi(W_0)$ is a minimal zero-sum sequence. Assuming this fails, then W_0 contains disjoint, nontrivial subsequences $W_{2m-1+k_m} \cdot W_{2m+k_m} \mid W_0$ with $|W_{2m-1+k_m}| + |W_{2m+k_m}| \le n + 2n - 1$ and $\varphi(W_{2m-1+k_m})$ and $\varphi(W_{2m+k_m})$ both zero-sum (if $|W_0| \le 3n-1$, this is trivial in view of $\varphi(W_0)$ not being a minimal zero-sum, while the same conclusion follows from $\mathsf{D}(\varphi(G)) = 2n-1$ and $\mathsf{s}_{\le n}(\varphi(G)) = 3n-2$ when $|W_0| \ge 3n-1$). By passing to appropriate zero-sum subsequences, we can then further assume $\varphi(W_{2m-1+k_m})$ and $\varphi(W_{2m+k_m})$ are each minimal zero-sum subsequences, so that $|W_i| \le \mathsf{D}(\varphi(G)) = 2n-1$ for both $i \in \{2m-1+k_m, 2m-k_m\}$. As at most one of the sequences W_j can contain the term g_0, it follows that $\prod^\bullet_{i\in[1,2m+k_m]\setminus\{j\}} W_j \mid S$ for some $j \in [1, 2m+k_m]$. Now $\sigma(W_1) \cdot \ldots \cdot \sigma(W_{2m+k_m}) \cdot \sigma(W_j)^{[-1]}$ is a sequence of terms from $\ker \varphi \cong C_m \oplus C_m$ with length $\mathsf{s}_{\le 2m-1-k_m}(C_m \oplus C_m) = 2m-1+k_m$. It follows that there is a zero-sum subsequence $\prod^\bullet_{i\in I} \sigma(W_i)$ for some $I \subseteq [1, 2m+k_m] \setminus \{j\}$ with $1 \le |I| \le 2m-1-k_m$. In such case, if $|I| \ge 2$, then $T = \prod^\bullet_{i\in I} W_i$ is a zero-sum subsequence of S with length

$$\begin{aligned} |T| &\le (|I|-2)n + \max\{2n, n + |W_{2m-1+k_m}|, n + |W_{2m+k_m}|, |W_{2m-1+k_m}| + |W_{2m+k_m}|\} \\ &\le (|I|-2)n + 3n - 1 \le (2m-3-k_m)n + 3n - 1 = 2mn - 1 + k, \end{aligned}$$

contrary to (6). On the other hand, if $|I|=1$, then $T=\prod^\bullet_{i\in I} W_i$ is a zero-sum subsequence of S with length $|T|\leq \max\{n, |W_{2m-1+k_m}|, |W_{2m+k_m}|\}\leq 2n-1\leq 2mn-1-k$, also contradicting (6). This shows that $\varphi(W_0)$ must be a minimal zero-sum sequence. In particular,

$$|W_0|\leq \mathsf{D}(C_n\oplus C_n)=2n-1.$$

Regardless of whether $k_n=0$ or 1, we have shown that $|W_0|\leq 2n-1$. As a result, since $|W_i|\leq n$ for all $i\in[1,2m-2+k_m]$, we have

$$2n-1=2mn-1+k_mn-(2m-2+k_m)n\leq |S^*|-\sum_{i=1}^{2m-2+k_m}|W_i|=|W_0|\leq 2n-1,$$

forcing equality to hold in these estimates, i.e., $|W_i|=n$ for all $i\in[1,2m-2+k_m]$ and $|W_0|=2n-1$. If $k_n=0$, we have already shown that $\varphi(W_0)$ is a minimal zero-sum sequence. For $k_n=1$, we established that $0\notin\Sigma_{2n-2}(\varphi(W_0))$, which combined with $\mathsf{D}(\varphi(G))=\mathsf{D}(C_n\oplus C_n)=2n-1$ forces $\varphi(W_0)$ to be a minimal zero-sum sequence in this case as well. If $0\in\Sigma_{\leq n-1}(\varphi(S^*))$, then there is a nontrivial subsequence $W_1'\mid S^*$ with $1\leq |W_1'|\leq n-1$ and $\varphi(W_1')$ zero-sum. Then, by the argument showing that S^* has some block decomposition, we can find a block decomposition $S^*=W_0'\boldsymbol{\cdot}W_1'\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}W_{2m-2+k_m}'$ with $|W_1'|<n$, contrary to what was just established for an arbitrary block decomposition. Thus $0\notin\Sigma_{\leq n-1}(\varphi(S^*))$, and all parts of Claim A are established. □

Suppose

$$S^*=W_0\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}W_{2m-2+k_m} \tag{7}$$

with each $\varphi(W_i)$ a nontrivial zero-sum for $i\in[0,2m-2+k_m]$. We call this a *weak block decomposition* of S^*. In view of Claim A, we have $|W_i|\geq n$ for all $\in[0,2m-2+k_m]$, and since $|S^*|=2mn-1-k_mn>(2m-1-k_m)n$, we cannot have $|W_i|=n$ for all $i\in[0,2m-2+k_m]$.

$$\text{Let } k_\emptyset\in[0,2m-2+k_m] \text{ be an index with } \quad \begin{cases} |W_{k_\emptyset}|>n & \text{if } k_n=1,\\ g_0\in\operatorname{Supp}(W_0) & \text{if } k_n=0.\end{cases}$$

Then define

$$S_\sigma=\sigma(W_0)\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}\sigma(W_{2m-2+k_m})\boldsymbol{\cdot}\sigma(W_{k_\emptyset})^{[-1]}\in\mathcal{F}(\ker\varphi).$$

We call $k_\emptyset$ and S_σ the *associated* index and sequence for the block decomposition. For $j\in[0,2m-2+k_m]$, set

$$\widetilde{W}_j=\begin{cases} W_j\boldsymbol{\cdot}g_0^{[-1]} & \text{if } k_n=0 \text{ and } j=k_\emptyset;\\ W_j & \text{otherwise.}\end{cases}$$

In view of Claim A, any block decomposition is also a weak block decomposition. If $S^*=W_0\boldsymbol{\cdot}W_1\boldsymbol{\cdot}\ldots\boldsymbol{\cdot}W_{2m-2+k_m}$ is a block decomposition and $k_n=1$, then $k_\emptyset=0$ is forced as

$|W_i| = n$ for all $i \geq 1$. On the other hand, if $k_n = 0$, then there is a block decomposition with $g_0 \in \mathrm{Supp}(W_0)$ as remarked earlier, and thus with $k_\emptyset = 0$. For $j \in [0, 2m-2+k_m]$, set

$$\widetilde{W}_j = \begin{cases} g_0^{-1} \cdot W_j & \text{if } k_n = 0 \text{ and } j = k_\emptyset, \\ W_j & \text{otherwise.} \end{cases}$$

Claim B. Suppose $S^* = W_0 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} W_{2m-2+k_m}$ is a weak block decomposition with associated index $k_\emptyset$ and associated sequence S_σ. Then $|S_\sigma| = 2m-2+k_m$ with $0 \notin \Sigma_{\leq 2m-1-k_m}(S_\sigma)$. Moreover, if we also have $k_n = 1$, then $|\sigma(W_{k_\emptyset}) \boldsymbol{\cdot} S_\sigma| = 2m-1+k_m$ with $0 \notin \Sigma_{\leq 2m-2-k_m}(\sigma(W_{k_\emptyset}) \boldsymbol{\cdot} S_\sigma)$. Regardless of whether $k_n = 0$ or 1, Conjecture 1.1 holds for S_σ.

Proof. We have $|S_\sigma| = 2m-2+k_m$ by definition. Assume by contradiction $0 \in \Sigma_{\leq 2m-1-k_m}(S_\sigma)$. Then there is a zero-sum subsequence $\prod^\bullet_{i\in I} \sigma(W_i)$ for some $I \subseteq [0, 2m-2+k_m] \setminus \{k_\emptyset\}$ with $1 \leq |I| \leq 2m-1-k_m$. By Claim A, we have $0 \notin \Sigma_{\leq n-1}(\varphi(S^*))$, which ensures $|W_i| \geq n$ for all $i \in [0, 2m-2+k_m] \setminus I$. Since $k_\emptyset \notin I$, we have $|W_{k_\emptyset}| \geq n+k_n$ with $k_\emptyset \in [0, 2m-2+k_m] \setminus I$ (by definition of $k_\emptyset$). It follows that $T := \prod^\bullet_{i\in I} W_i$ is a nontrivial zero-sum subsequence of S with length $|T| = |S^*| - \sum_{i\in[0,2m-2+k_m]\setminus I} |W_i| \leq |S^*| - (2m-1+k_m-|I|)n - k_n \leq |S^*| - 2k_m n - k_n = 2mn-1-k$, contradicting (6). So we instead conclude that $|S_\sigma| = 2m-2+k_m$ and $0 \notin \Sigma_{\leq 2m-1-k_m}(S_\sigma)$.

Suppose $k_n = 1$, so that $S^* = S$. Assume by contradiction that $0 \in \Sigma_{\leq 2m-2-k_m}(\sigma(W_{k_\emptyset}) \boldsymbol{\cdot} S_\sigma)$. Then there is a zero-sum subsequence $\prod^\bullet_{i\in I} \sigma(W_i)$ for some $I \subseteq [0, 2m-2+k_m]$ with $1 \leq |I| \leq 2m-2-k_m$. By Claim A, we have $0 \notin \Sigma_{\leq n-1}(\varphi(S^*))$, which ensures $|W_i| \geq n$ for all $i \in [0, 2m-2+k_m] \setminus I$. Hence $T := \prod^\bullet_{i\in I} W_i$ is a nontrivial zero-sum subsequence of S with length

$$\begin{aligned} |T| = |S^*| - \sum_{i\in[0,2m-2+k_m]\setminus I} |W_i| &\leq |S^*| - (2m-1+k_m-|I|)n \leq |S^*| - (2k_m+1)n \\ &= 2mn - 1 - k_m n - n \leq 2mn-1-k, \end{aligned}$$

contradicting (6). So we instead conclude that we have $|\sigma(W_{k_\emptyset}) \boldsymbol{\cdot} S_\sigma| = 2m-1+k_m$ and $0 \notin \Sigma_{\leq 2m-2-k_m}(\sigma(W_{k_\emptyset}) \boldsymbol{\cdot} S_\sigma)$.

If Conjecture 1.1 holds for k_m in $C_m \oplus C_m$, then the first part of Claim B ensures that Conjecture 1.1 holds for S_σ. Otherwise, the hypotheses of Proposition 3.1 ensure that $k_n = 1$ and $k_m \in [1, m-2]$ with Conjecture 1.1 holding for k_m+1 in $C_m \oplus C_m$. In such case, the second part of Claim B ensures that Conjecture 1.1 holds for $\sigma(W_{k_\emptyset}) \boldsymbol{\cdot} S_\sigma$, and then applying Lemma 2.4 shows that Conjecture 1.1 holds for S_σ. □

Claim C. There exists a basis (f_1, f_2) for $\varphi(G) = C_n \oplus C_n$ such that either

1. $\mathrm{Supp}(\varphi(S^*)) \subseteq f_1 \cup (\langle f_1 \rangle + f_2)$, or
2. $\varphi(S^*) = f_1^{[an]} \boldsymbol{\cdot} f_2^{[bn-1]} \boldsymbol{\cdot} f_3^{[cn-1]} \boldsymbol{\cdot} (f_2+f_3)$, where $f_3 = xf_1 + f_2$ for some $x \in [2, n-2]$ with $\gcd(x, n) = 1$, $a, b, c \geq 1$ and $a+b+c = 2m+k_m$.

Proof. By Claim A, we have $0 \notin \Sigma_{\leq n-1}(\varphi(S^*))$, while $|\varphi(S^*)| = |S^*| = (2m+k_m)n - 1$. Thus Claim C follows from Theorem B. □

We define a term $g \in \mathrm{Supp}(S)$ to be *good* if $g, h \in \mathrm{Supp}(S)$ with $\varphi(g) = \varphi(h)$ implies $g = h$. A term $g \in \mathrm{Supp}(\varphi(S))$ is *good* if $\mathrm{Supp}(S)$ contains exactly one element from $\varphi^{-1}(g)$. Then, for $g \in \mathrm{Supp}(S)$, we find that $\varphi(g) = mg$ is good if and only if g is good.

Claim D. Suppose $S^* = W_0 \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ is a weak block decomposition. If $g \in \mathrm{Supp}(W_j)$, $h \in \mathrm{Supp}(\widetilde{W}_{k_\emptyset})$ and $\varphi(g) = \varphi(h)$, where $j, k_\emptyset \in [0, 2m-2+k_m]$ are *distinct*, then $g = h$ is good.

Proof. Since $\varphi(g) = \varphi(h)$, setting $W'_j = W_j \cdot g^{[-1]} \cdot h$, $W'_{k_\emptyset} = W_{k_\emptyset} \cdot h^{[-1]} \cdot g$, and $W'_i = W_i$ for all $i \neq j, k$, we obtain a new weak block decomposition $S^* = W'_0 \cdot W'_1 \cdot \ldots \cdot W'_{2m-2+k_m}$. Since $h \in \mathrm{Supp}(\widetilde{W}_{k_\emptyset})$, we have $g_0 \in \mathrm{Supp}(W'_{k_\emptyset})$ for $k_n = 0$, and $|W'_{k_\emptyset}| = |W_{k_\emptyset}| > n$ for $k_n = 1$. Consequently, if we let S_σ and S'_σ be the associated sequences for the original and new block decompositions, with $k_\emptyset$ and $k'_\emptyset$ the associated indices, we find that $k_\emptyset = k'_\emptyset$ with S'_σ obtained from S_σ by replacing the term $\sigma(W_j)$ by the term $\sigma(W'_j) = \sigma(W_j) - g + h$. In view of Claim B, it follows that Conjecture 1.1 holds for both sequences S'_σ and S_σ using $k_m \in [1, m-1]$ modulo m. Thus Lemma 2.1 implies that $g = h$. Repeating this argument for an arbitrary $g' \in \mathrm{Supp}(S \cdot W_{k_\emptyset}^{[-1]})$ using the fixed $h \in \mathrm{Supp}(\widetilde{W}_{k_\emptyset})$, we conclude that $g' = h = g$ for all $g' \in \mathrm{Supp}(S \cdot W_{k_\emptyset}^{[-1]})$ with $\varphi(g') = \varphi(g) = \varphi(h)$. Likewise, repeating the argument for an arbitrary $h' \in \mathrm{Supp}(\widetilde{W}_{k_\emptyset})$ using the fixed $g \in \mathrm{Supp}(W_j)$, we find $h' = g = h$ for all $h' \in \mathrm{Supp}(\widetilde{W}_{k_\emptyset})$ with $\varphi(h') = \varphi(g) = \varphi(h)$. It follows that $g = h$ is good. □

Claim E. Claim C.1 holds for S^*.

Proof. Assume instead that Claim C.2 holds. Let (f_1, f_2) be a basis for which Claim C.2 holds and let $x^* \in [2, n-2]$ be the multiplicative inverse of $-x$ modulo n, so

$$x^* x \equiv -1 \mod n.$$

In view of Claim C.2, there is a block decomposition $S^* = W_0 \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ with $\varphi(W_0) = f_1^{[n-1]} \cdot f_2^{[x^*]} \cdot f_3^{[n-x^*]}$, $\varphi(W_1) = f_3^{[x^*-1]} \cdot f_2^{[n-x^*-1]} \cdot f_1 \cdot (f_2 + f_3)$ and $\varphi(W_i) \in \{f_1^{[n]}, f_2^{[n]}, f_3^{[n]}\}$ for all $i \in [2, 2m-2+k_m]$. We call any such block decomposition a *strong* block decomposition of S^*. For $k_n = 0$, it can also be assumed that either $g_0 \in \mathrm{Supp}(W_0)$ or else $g_0 \in \mathrm{Supp}(W_1)$ and $\varphi(g_0) = f_2 + f_3$, since $\mathrm{Supp}(\varphi(W_0))$ contains every element from $\mathrm{Supp}(\varphi(S^*))$ apart from the unique term equal to $f_2 + f_3$ which is contained in $\varphi(W_1)$.

Let us first show all $g \in \mathrm{Supp}(S)$ are good. Since Claim C.2 holds, we have $n \geq 5$ (as $x \in [2, n-2]$ with $\gcd(x, n) = 1$) and there is a unique term g of S^* with $\varphi(g) = f_2 + f_3$, which is trivially good if $g \in \mathrm{Supp}(S)$. Consider $f \in \{f_1, f_2, f_1 + f_2\}$ and let $S^* = W_0 \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ be a strong block decomposition, and if $k_n = 0$, assume either $g_0 \in \mathrm{Supp}(W_0)$ or else $g_0 \in \mathrm{Supp}(W_1)$ and $\varphi(g_0) = f_2 + f_3$. Since $\mathsf{v}_f(\varphi(W_0)) \geq 2$, there is some $h \in \mathrm{Supp}(\widetilde{W}_0)$ with

$\varphi(h) = f$. Since $\mathsf{v}_f(\varphi(W_1)) \geq 1$ with either $g_0 \in \text{Supp}(W_0)$ or $\varphi(g_0) = f_2 + f_3$, there is some $g \in \text{Supp}(\widetilde{W}_1)$ with $\varphi(g) = f$. If $k_n = 1$, then $k_\emptyset = 0$, and if $k_n = 0$, then $k_\emptyset \in \{0, 1\}$ (as $g_0 \in \text{Supp}(W_0 \cdot W_1)$). Thus applying Claim D shows that $\varphi(g) = f$ is good, as claimed.

Now, since all $g \in \text{Supp}(S)$ are good, an appropriate choice of pre-images for the elements f_1 and f_2 yields $\text{Supp}(S) \subseteq \{e_1, e_2, e_3 + \alpha, e_2 + e_3 + \beta\}$ for some $\alpha, \beta \in \ker \varphi$, where $e_3 := xe_1 + e_2$, $\varphi(e_1) = f_1$ and $\varphi(e_2) = f_2$. As before, let $S^* = W_0 \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ be a strong block decomposition, and if $k_n = 0$, assume either $g_0 \in \text{Supp}(W_0)$ or else $g_0 \in \text{Supp}(W_1)$ and $\varphi(g_0) = f_2 + f_3$. By choosing $g_0 \in g_0 + \ker \varphi$ appropriately, we can assume $g_0 \in \{e_1, e_2, e_3 + \alpha, e_2 + e_3 + \beta\}$ for $k_n = 0$ as well. Since $x, x^* \in [2, n-2]$, we have $\mathsf{v}_{e_1}(W_0) = n - 1 > x$ and $\mathsf{v}_{e_2}(W_0) = x^* > 1$, whence $e_1^{[x]} \cdot e_2 \mid \widetilde{W}_0$ and $e_3 + \alpha \mid \widetilde{W}_1$. Thus, setting $W_0' = W_0 \cdot (e_1^{[x]} \cdot e_2)^{[-1]} \cdot (e_3 + \alpha)$, $W_1' = W_1 \cdot (e_3 + \alpha)^{[-1]} \cdot e_1^{[x]} \cdot e_2$ and $W_i' = W_i$ for $i \geq 2$, we obtain a weak block decomposition $S^* = W_0' \cdot W_1' \cdot \ldots \cdot W_{2m-2+k_m}'$ say with associated index $k_\emptyset'$ and associated sequence S_σ'. Since $e_1^{[x]} \cdot e_2 \mid \widetilde{W}_0$ and $e_3 + \alpha \mid \widetilde{W}_1$, we have $k_\emptyset' = k_\emptyset \in \{0, 1\}$ when $k_n = 0$, while $|W_0'| = 2n - 1 - x \geq n + 1$ ensures that $k_\emptyset' = k_\emptyset = 0$ for $k_n = 1$. As a result, Claim B and Lemma 2.1 imply $S_\sigma = S_\sigma'$ with $\alpha = 0$. Similarly, since $x, x^* \in [2, n-2]$, we have $\mathsf{v}_{e_1}(W_0) = n - 1 > n - x$ and $\mathsf{v}_{e_3}(W_0) = n - x^* > 1$, whence $e_1^{[n-x]} \cdot e_3 \mid \widetilde{W}_0$ and $e_2 \mid \widetilde{W}_1$. Setting $W_0' = W_0 \cdot (e_1^{[n-x]} \cdot e_3)^{[-1]} \cdot e_2$, $W_1' = W_1 \cdot e_2^{[-1]} \cdot e_1^{[n-x]} \cdot e_3$ and $W_i' = W_i$ for $i \geq 2$, we obtain a weak block decomposition $S^* = W_0' \cdot W_1' \cdot \ldots \cdot W_{2m-2+k_m}'$ with $|W_0'| = 2n - 1 - (n - x) \geq n + 1$ having associated index $k_\emptyset' = k_\emptyset$ and associated sequence S_σ'. Thus Claim B and Lemma 2.1 yield $S_\sigma = S_\sigma'$ and $ne_1 = (n - x)e_1 + e_3 - e_2 = 0$. We thus obtain a zero-sum subsequence $e_1^{[n]} \mid S$, contradicting that $0 \notin \Sigma_{\leq n}(S)$ (which holds by (6)), unless $a = 1$, $k_n = 0$ and $\varphi(g_0) = f_1$. However, in such case, there are $|S^*| - n - 1 = (2m + k_m - 1)n - 2 \geq 2mn - 2$ terms of S equal to either e_2 or e_3, so that the pigeonhole principle yields that either e_2 or e_3 has multiplicity at least $mn - 1$ in S. If either has multiplicity at least mn, then $e_2^{[mn]}$ or $e_3^{[mn]}$ is a zero-sum subsequence of length mn, contradicting that $0 \notin \Sigma_{\leq mn}(S)$ by (6). On the other hand, if both e_2 and $e_3 = xe_1 + e_2$ have multiplicity $mn - 1 \geq n$ (as $m \geq 2$), then $e_2^{[mn-n]} \cdot e_3^{[n]}$ is a zero-sum subsequence of length mn (as $ne_1 = 0$), again contradicting that $0 \notin \Sigma_{\leq mn}(S)$. □

In view of Claim E, we now assume Claim C.1 holds for the remainder of the proof, say with basis (f_1, f_2). Then $\text{Supp}(\varphi(S^*)) \subseteq f_1 \cup \big(\langle f_1 \rangle + f_2\big)$, implying that the number of terms from $\langle f_1 \rangle + f_2$ in any zero-sum subsequence of $\varphi(S)$ must be congruent to 0 modulo n. As a result, if $S^* = W_0 \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ is any block decomposition, then since $|W_i| = n$ for $i \geq 1$ and $|W_i| = 2n - 1$, we find that

$$(8)\quad \varphi(W_0) = f_1^{[n-1]} \cdot {\prod}^{\bullet}_{i \in [1,n]} (x_i f_1 + f_2) \quad \text{and} \quad \varphi(W_i) \in \{f_1^{[n]}, {\prod}^{\bullet}_{i \in [1,n]} (c_i f_1 + f_2)\} \text{ for } i \geq 1,$$

where $x_1, \ldots, x_n, c_1, \ldots, c_n \in [0, n-1]$ with $c_1 + \ldots + c_n \equiv 0 \mod n$ and $x_1 + \ldots + x_n \equiv 1 \mod n$. Note, if (f_1, f_2) is a basis for which Claim C.1 holds, then so is $(f_1, xf_1 + f_2)$ for any $x \in \mathbb{Z}$.

Claim F. If $n = 2$, then all terms $x \in \text{Supp}(S)$ are good.

Proof. Let $S^* = W_0 \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ be a block decomposition. Then $\varphi(W_0)$ is a minimal zero-sum sequence of length $2n - 1 = 3$ by Claim A, so $\varphi(W_0) = f_1 \cdot f_2 \cdot (f_1 + f_2)$ with f_1, f_2 and $f_1 + f_2$ the three nonzero elements of $\varphi(G) \cong C_n \oplus C_n = C_2 \oplus C_2$. By Claim C.1, we have $\text{Supp}(\varphi(S)) \subseteq \{f_1, f_2, f_1 + f_2\} = \varphi(G) \setminus \{0\}$, and $|\text{Supp}(\varphi(W_i))| = 1$ for $i \in [1, 2m - 2 + k_m]$, allowing us to assume $g_0 \in \text{Supp}(W_0)$ when $k_n = 0$, and by choosing f_1 and f_2 appropriately, we can w.l.o.g. assume $\varphi(g_0) = f_1 + f_2$. If $\mathsf{v}_{f_1}(S) = 1$, then the definition of good holds trivially for f_1. Otherwise, there is some W_j with $f_1 \in \text{Supp}(W_j)$ and $j \geq 1$, in which case Claim D, implies that f_1 is good. Thus f_1 is good, and the same argument shows that f_2 is good. If $k_n = 1$, the argument also shows $f_1 + f_2$ is good. The proof of the claim is now complete unless $\mathsf{v}_{f_1+f_2}(\varphi(S)) \geq 2$ with $k_n = 0$ and $S^* = S \cdot g_0$, which we now assume. Note $\varphi(g_0) = f_1 + f_2$, so $W'_1 = (f_1 + f_2) \cdot \varphi(g_0)$ is a length two zero-sum dividing $\varphi(S \cdot g_0)$. Applying the argument showing the existence of a block decomposition, it follows that there is a block decomposition $W'_0 \cdot W'_1 \cdot \ldots \cdot W'_{2m-2+k_m}$ of S^* with $\varphi(W'_1) = (f_1 + f_2) \cdot \varphi(g_0)$ and $k_\emptyset = 1$. Since $\mathsf{v}_{f_1+f_2}(\varphi(S)) \geq 2$, it follows that there is some $j \in [0, 2m - 2 + k_m] \setminus \{1\}$ with $f_1 + f_2 \in \text{Supp}(\varphi(W_j))$, and now Claim D, applied to the block decomposition $W'_0 \cdot W'_1 \cdot \ldots \cdot W'_{2m-2+k_m}$, implies that $f_1 + f_2$ is good, completing the claim. □

Claim G. Suppose $S^* = W_0 \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ is a block decomposition with $k_\emptyset = 0$. If $g_1 \in \text{Supp}(\widetilde{W}_{j_1})$, $g_2 \in \text{Supp}(\widetilde{W}_{j_2})$ and $\varphi(g_1) = \varphi(g_2)$, where j_1, $j_2 \in [0, 2m-2+k_m]$ are *distinct*, then $g_1 = g_2$ is good.

Proof. In view of Claim F, we can assume $n \geq 3$. Since $\mathsf{v}_{f_1}(\varphi(W_0)) = n - 1 \geq 2$ and $|W_0| - \mathsf{v}_{f_1}(\varphi(W_0)) = n \geq 3$ by (8), we have and can w.l.o.g. assume (by possibly exchanging f_2 for an appropriate alternative from $\langle f_1 \rangle + f_2$) that

$$f_1 \in \text{Supp}(\varphi(\widetilde{W}_0)) \quad \text{and} \quad f_2 \in \text{Supp}(\varphi(\widetilde{W}_0)). \tag{9}$$

If $j_1 = 0$ or $j_2 = 0$, then Claim D implies $g_1 = g_2$ is good (as $k_\emptyset = 0$). Therefore we can w.l.o.g. assume $j_1 = 1$ and $j_2 = 2$. By Claim C.1, we have $\text{Supp}(\varphi(S)) \subseteq \{f_1\} \cup \big(\langle f_1 \rangle + f_2\big)$. If $\varphi(g_1) = f_1 \in \text{Supp}(\varphi(\widetilde{W}_0))$, then Claim D applied with $k_\emptyset = 0$ and $j = 1$ implies that g_1 is good. Therefore, in view of (8), we can assume

$$\text{Supp}(\varphi(W_1)) \subseteq \langle f_1 \rangle + f_2 \quad \text{and} \quad \text{Supp}(\varphi(W_2)) \subseteq \langle f_1 \rangle + f_2, \tag{10}$$

with the latter following by an analogous argument. In particular,

$$\varphi(g_1) = \varphi(g_2) = x_1 f_1 + f_2 \quad \text{for some } x_1 \in [0, n-1].$$

Suppose $k_n = 1$. Then $|W_0 \cdot W_1 \cdot g_1^{[-1]}| = 3n - 2 = \eta(C_n \oplus C_n)$, ensuring by Claim A that $W_0 \cdot W_1 \cdot g_1^{[-1]}$ contains an n-term subsequence W'_0 with $\varphi(W'_0)$ zero-sum. Setting $W'_0 = W_0 \cdot W_1 \cdot (W'_1)^{[-1]}$, it follows that $S^* = W'_0 \cdot W'_1 \cdot W_2 \cdot \ldots \cdot W_{2m-2+k_m}$ is a block decomposition with $g_1 \in \mathrm{Supp}(W_0')$, $g_2 \in \mathrm{Supp}(W_2')$ and associated index $k_\emptyset' = 0$, in which case Claim D implies that $g_1 = g_2$ is good, as desired.

Suppose $k_n = 0$. Since $k_\emptyset = 0$, we have $g_0 \in \mathrm{Supp}(W_0)$ with $|W_0 \cdot W_1 \cdot g_1^{[-1]} \cdot g_0^{[-1]}| = 3n-3$. If $W_0 \cdot W_1 \cdot g_1^{[-1]} \cdot g_0^{[-1]}$ contains an n-term subsequence W_1' with $\varphi(W_0')$ zero-sum, then setting $W_0' = W_0 \cdot W_1 \cdot (W_1')^{[-1]}$, it follows that $S^* = W_0' \cdot W_1' \cdot W_2 \cdot \ldots \cdot W_{2m-2+k_m}$ is a block decomposition with $g_0 \cdot g_1 \mid W_0'$, $g_2 \in \mathrm{Supp}(W_2)$ and associated index $k_\emptyset' = 0$, in which case Claim D implies that $g_1 = g_2$ is good. Therefore, in view of Claim A, we can instead assume $0 \notin \Sigma_{\le n}(\varphi(W_0 \cdot W_1 \cdot g_1^{[-1]} \cdot g_0^{[-1]}))$. We have $f_1,\, f_2 \in \mathrm{Supp}(\varphi(W_0 \cdot g_0^{[-1]})) \subseteq \mathrm{Supp}(W_0')$ by (9), so applying the established Conjecture 1.1.4 to $\varphi(W_0 \cdot W_1 \cdot g_1^{[-1]} \cdot g_0^{[-1]})$ yields

$$\varphi(W_0 \cdot W_1 \cdot g_1^{[-1]} \cdot g_0^{[-1]}) = f_1^{[n-1]} \cdot f_2^{[n-1]} \cdot f_3^{[n-1]} \quad \text{for some } f_3 = x_3 f_1 + f_2, \tag{11}$$

where we have $f_3 = x_3 f_1 + f_2$ since $\mathrm{Supp}(\varphi(S)) \subseteq \{f_1\} \cup \big(\langle f_1 \rangle + f_2\big)$. Moreover, since (f_2, f_3) must be a basis, it follows that

$$\gcd(x_3, n) = 1.$$

By (11), we have

$$\varphi(g_0) + \varphi(g_1) = -\sigma(\varphi(W_0 \cdot W_1 \cdot g_0^{[-1]} \cdot g_1^{[-1]})) = f_1 + f_2 + f_3 = (1 + x_3) f_2 + 2 f_2. \tag{12}$$

Observe that $\Sigma_{n-2}(f_2^{[n-2]} \cdot f_3^{[n-2]}) = \{x f_1 - 2 f_2 \;:\; x \in \underbrace{\{0, x_3\} + \ldots + \{0, x_3\}}_{n-2}\}$. Since $\gcd(x_3, n) = 1$, it follows that $\underbrace{\{0, x_3\} + \ldots + \{0, x_3\}}_{n-2}$ contains all residue classes modulo n except $(n-1)x_3 \equiv -x_3 \mod n$. As a result, since $-1 - x_3 \not\equiv -x_3 \mod n$, it follows from (12) that $-\varphi(g_0) - \varphi(g_1) \in \Sigma_{n-2}(f_2^{[n-2]} \cdot f_3^{[n-2]})$, which means (recall (11)) that there is an n-term subsequence $W_1' \mid W_0 \cdot W_1$ with $g_0 \cdot g_1 \mid W_1'$ and $\varphi(W_1')$ zero-sum. Letting $W_0' = W_0 \cdot W_1 \cdot (W_1')^{[-1]}$, it follows that $S^* = W_0' \cdot W_1' \cdot W_2 \cdot \ldots \cdot W_{2m-2+k_m}$ is a block decomposition with $g_1 \in \mathrm{Supp}(W_1' \cdot g_0^{[-1]})$ and $g_2 \in \mathrm{Supp}(W_2)$. It now follows from Claim D, applied to this block decomposition with $k_\emptyset = 1$ and $j = 2$, that $g_1 = g_2$ is good, as desired. □

CASE 1. $n = 2$.

In this case, $\mathrm{Supp}(\varphi(S^*)) \subseteq \{f_1\} \cup \big(\langle f_1 \rangle + f_2\big) = \{f_1, f_2, f_1 + f_2\}$ with f_1, f_2 and $f_1 + f_2$ the three nonzero elements of $\varphi(G) = C_n \oplus C_n = C_2 \oplus C_2$. Claim F ensures that all terms of S are good, so (choosing $g_0 \in g_0 + \ker \varphi$ appropriately when $k_n = 0$) we find

$$\mathrm{Supp}(S^*) = \{e_1, e_2, e_1 + e_2 + \alpha\}$$

for some $e_1, e_2, \alpha \in G$ with

$$m e_1 = \varphi(e_1) = f_1, \quad m e_2 = \varphi(e_2) = f_2 \quad \text{and} \quad m\alpha = \varphi(\alpha) = 0.$$

Let $S^* = W_0 \cdot \ldots \cdot W_{2m-2+k_m}$ be a block decomposition with $k_\emptyset = 0$ having associated sequence $S_\sigma = \prod^\bullet_{i \in [1, 2m-2+k_m]} \sigma(W_i)$. In view of (8), we have $W_0 = e_1 \cdot e_2 \cdot (e_1 + e_2 + \alpha)$ and $W_i \in \{e_1^{[2]}, e_2^{[2]}, (e_1 + e_2 + \alpha)^{[2]}\}$ for $i \ge 1$. Thus each term in S_σ is either equal to $2e_1$, $2e_2$ or $2e_1 + 2e_2 + 2\alpha$. In view of Claim B, we know Conjecture 1.1 holds for S_σ. Since *all* bases for

$\varphi(G) = C_n \oplus C_n$ satisfy Claim C.1 when $n = 2$, we can replace the basis (f_1, f_2) by any alternative one. Thus we can w.l.o.g. assume Conjecture 1.1 holds for S_σ using the basis $(2e_1, 2e_2)$ with $\mathrm{Supp}(S_\sigma) = \{2e_1, 2e_2, 2e_1 + 2e_2 + 2\alpha\}$.

Suppose $k_m = 1 < m - 1$. Then $m \geq 3$, $k = k_m n + k_n \in \{2, 3\}$, and Conjecture 1.1 implies that $2e_1$ occurs with multiplicity $m - 1$ in S_σ, $2e_2$ occurs with multiplicity $x \geq 1$ in S_σ, $2e_1 + 2e_2 + 2\alpha$ occurs with multiplicity $m - x \geq 1$ in S_σ, and $(2e_1 + 2e_2 + 2\alpha) - 2e_2 \in \langle 2e_1 \rangle$, whence $e_1 + e_2 + \alpha = ye_1 + (1 + m)e_2$ or $ye_1 + e_2$ for some $y \in [0, 2m - 1]$. We can assume the latter does not occur, else Lemma 2.2 yields the desired structure for S. Hence, by swapping the basis (f_1, f_2) with $(f_1, f_1 + f_2)$ if need be, we can assume $x \geq m - x$ and

$$S^* = e_1^{[2m-1]} \boldsymbol{\cdot} e_2^{[2x+1]} \boldsymbol{\cdot} (ye_1 + (1 + m)e_2)^{[2(m-x)+1]}$$

for some $x \in [\frac{m}{2}, m - 1]$ and $y \in [0, 2m - 1]$. If $y = 0$, then $e_2^{[m-1]} \boldsymbol{\cdot} (1 + m)e_2$ is a zero-sum subsequence of S with length m, contrary to (6). Therefore $y \geq 1$. If $y \geq 2$, or $k_n = 1$, or $k_n = 0$ with $\varphi(g_0) \neq f_1$, then $e_1^{[2m-y]} \boldsymbol{\cdot} e_2^{[m-1]} \boldsymbol{\cdot} (ye_1 + (1 + m)e_2)$ is a nontrivial zero-sum subsequence of S with length $3m - y \leq 3m - 1 \leq 4m - 4 \leq 2mn - 1 - k$, contradicting (6). On the other hand, if $y = 1$, $k_n = 0$ and $\varphi(g_0) = f_1$, then $e_1^{[2m-3]} \boldsymbol{\cdot} e_2^{[m-3]} \boldsymbol{\cdot} (e_1 + (1 + m)e_2)^{[3]}$ is a nontrivial zero-sum subsequence of S with length $3m - 3 \leq 4m - 4 \leq 2mn - 1 - k$, again contradicting (6). So we can now assume either $k_m \in [2, m - 1]$ or $m = 2$.

In this case, Conjecture 1.1 holding for S_σ with basis $(2e_1, 2e_2)$ means $2e_1$ and $2e_2$ occur with multiplicity $m - 1$ in S_σ, $2e_1 + 2e_2 + 2\alpha$ occurs with multiplicity k_m in S_σ,

$$\langle 2e_1 + 2\alpha \rangle = \langle 2e_1 \rangle, \quad \text{and either} \quad 2\alpha = 0 \ \text{ or } \ k_m = m - 1. \tag{13}$$

Moreover, both $2\alpha = 0$ and $k_m = m - 1 = 1$ when $m = 2$. It follows that

$$S^* = e_1^{[2m-1]} \boldsymbol{\cdot} e_2^{[2m-1]} \boldsymbol{\cdot} (e_1 + e_2 + \alpha)^{[2k_m+1]}.$$

If $H = \langle e_1, e_2 \rangle$ is a proper subgroup, then S contains a subsequence with two distinct terms from H and length at least $4m - 3 \geq \eta(H) - 1$, and thus contains a nontrivial zero-sum of length at most $\exp(H) \leq 2m$ using the established Conjecture 1.1.4, contrary to (6). Therefore $H = \langle e_1, e_2 \rangle = G$, forcing (e_1, e_2) to be a basis for $G = C_{2m} \oplus C_{2m}$. If $\alpha \in \langle e_1 \rangle$ or $\alpha \in \langle e_2 \rangle$, then Lemma 2.2 implies that S has the desired structure, completing the proof. Hence we may assume otherwise, so in view of $m\alpha = 0$ and $\langle 2e_1 + 2\alpha \rangle = \langle 2e_1 \rangle$, it follows that

$$\alpha = xe_1 + me_2 \quad \text{for some } x \in [1, 2m - 1]$$

with m even and

$$\mathrm{ord}(2(1 + x)e_1) = \mathrm{ord}(2e_1 + 2\alpha) = \mathrm{ord}(2e_1) = m.$$

We have two final subcases based upon which possibility occurs in (13).

Suppose $k_m = m - 1 \geq 2$. Then $e_1 + e_2 + \alpha = (1 + x)e_1 + (1 + m)e_2$. For $r \in [0, \frac{m}{2} - 1]$, we have $T_r := e_2^{[m-2r-1]} \boldsymbol{\cdot} ((1 + x)e_1 + (1 + m)e_2)^{[2r+1]}$ as a subsequence of S with sum $(1 + x)e_1 + r \cdot 2(1 + x)e_1$. Since $\mathrm{ord}(2(1 + x)e_1) = m$, it follows that $\{\sigma(T_r) : r \in [0, \frac{m}{2} - 1]\} \subseteq (1 + x)e_1 + [1, m]_{2e_1}$ is a subset of cardinality $\frac{m}{2}$, so there must be some $r \in [0, \frac{m}{2}-1]$ such that $\sigma(T_r) = ye_1$ for some $y \in [1, 2m]$ with $y \geq 2(\frac{m}{2}-1)+1 = m-1 \geq 2$. It follows that $e_1^{[2m-y]} \cdot T_r$ is a nontrivial zero-sum subsequence of S with length $3m - y \leq 2m+1$, contradicting (6) (as $k \leq 2m-2$).

Suppose $2\alpha = 0$. Then $e_1 + e_2 + \alpha = (1+m)e_1 + (1+m)e_2$, else Lemma 2.2 yields the desired structure for S. If $2k_m - 1 \leq m$, then $e_1^{[m-2k_m+1]} \cdot e_2^{[m-2k_m+1]} \cdot ((1+m)e_1 + (1+m)e_2)^{2k_m-1}$ is a nontrivial zero-sum subsequence of S with length $2m - 2k_m + 1 \leq 2m-1$, contradicting (6). On the other hand, if $2k_m - 1 \geq m+1$, then $e_1 \cdot e_2 \cdot ((1+m)e_1 + (1+m)e_2)^{m-1}$ is a nontrivial zero-sum subsequence of S with length $m+1 \leq 2m+1$, again contradicting (6) and completing the case.

CASE 2. $n \geq 3$.

Since $n \geq 3$, if $S^* = W_0 \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ is any block decomposition, then (8) ensures

$$f_1 \in \mathrm{Supp}(\varphi(\widetilde{W}_0)). \tag{14}$$

Claim H. The term $f_1 \in \mathrm{Supp}(\varphi(S))$ is good.

Proof. Let $S^* = W_0 \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ be a block decomposition with associated index $k_\emptyset = 0$. If $f_1 \in \mathrm{Supp}(\varphi(S \cdot W_0^{[-1]}))$, then (14) together with Claim D implies that f_1 is good. Therefore we can instead assume

$$\mathrm{Supp}(\varphi(S \cdot W_0^{[-1]})) \subseteq \langle f_1 \rangle + f_2. \tag{15}$$

We can also assume

$$\mathsf{v}_{f_1}(\varphi(S)) \geq 2, \tag{16}$$

lest f_1 being good holds trivially.

Let us show there are $g \in \mathrm{Supp}(\widetilde{W}_0)$ and $h \in \mathrm{Supp}(S \cdot W_0^{[-1]})$ with $\varphi(g)$, $\varphi(h) \in \langle f_1 \rangle + f_2$ and

$$\varphi(g) - \varphi(h) = zf_1 \quad \text{for some } z \in [2, n-1].$$

Assuming this fails, let $xf_1 + f_2 \in \mathrm{Supp}(\varphi(\widetilde{W}_0))$. Then (15) implies $\mathrm{Supp}(\varphi(S \cdot W_0^{[-1]})) \subseteq \{xf_1 + f_2, (x-1)f_1 + f_2\}$. If $\mathsf{v}_{xf_1+f_2}(\varphi(S)) \geq mn > n$, then the term g is good by Claim G, in turn implying that $\mathsf{v}_g(S) \geq mn$, where $g \in \mathrm{Supp}(S)$ is the unique term with $\varphi(g) = xf_1 + f_2$, whence $0 \in \Sigma_{mn}(S)$, contrary to (6). Therefore we can assume $\mathsf{v}_{xf_1+f_2}(\varphi(S)) \leq mn-1$, and thus $\mathsf{v}_{(x-1)f_1+f_2}(\varphi(S \cdot W_0^{[-1]})) \geq (2m-2+k_m)n - mn + 1 \geq mn - n + 1 > n$. Claim G now ensures that the term $(x-1)f_1 + f_2$ is also good, and thus has multiplicity at most $mn-1$ in $\varphi(S)$ lest we obtain the same contradiction as before. There are at least $(2m-2+k_m)n + n - 1 \geq 2mn - 1$ terms of $\varphi(S)$ from $\langle f_1 \rangle + f_2$. As a result, it follows that there is some $x'f_1 + f_2 \in \mathrm{Supp}(\varphi(\widetilde{W}_0))$ with $x'f_1 \notin \{xf_1, (x-1)f_1\}$. But now, taking $g' = x'f_1 + f_2$ and $h' = (x-1)f_1 + f_2 \in \mathrm{Supp}(\varphi(S \cdot W_0^{[-1]}))$, we find that $g' - h' = zf_1$ with $z \in [2, n-1]$, as desired. Thus the existence of g and h is established.

Let $j \in [1, 2m-2+k_m]$ be an index with $h \in \mathrm{Supp}(W_j)$. In view of (15) and (16), let $g_1 \cdot g_2 \mid \widetilde{W_0}$ be a length two subsequence with $\varphi(g_1) = \varphi(g_2) = f_1$. Since $1 \le n-z \le n-2$ and $\mathsf{v}_{f_1}(\varphi(W_0 \cdot g_2^{[-1]})) = n-2$, it follows that there is a subsequence $T \mid W_0 \cdot g_2^{[-1]}$ with $\varphi(T) = f_1^{[n-z]}$ and $g_1 \in \mathrm{Supp}(T)$. Since $\varphi(g) \in \langle f_1 \rangle + f_2$, we have $g \notin \mathrm{Supp}(T)$. Set

$$W_0' = W_0 \cdot T^{[-1]} \cdot g^{[-1]} \cdot h \qquad \text{and} \qquad W_j' = W_j \cdot h^{[-1]} \cdot g \cdot T$$

with $W_i' = W_i$ for $i \neq 0, j$. Then, by construction, $S^* = W_0' \cdot W_1' \cdot \ldots \cdot W_{2m-2+k_m}'$ is a weak block decomposition with associated index $k_\emptyset' \in \{0, j\}$. Moreover, $g_2 \in \mathrm{Supp}(\widetilde{W_0'})$ and $g_1 \in \mathrm{Supp}(\widetilde{W_j'})$ with $\varphi(g_1) = \varphi(g_2) = f_1$. Thus applying Claim D to $S^* = W_0' \cdot W_1' \cdot \ldots \cdot W_{2m-2+k_m}'$ implies f_1 is good, completing the claim. □

Let $e_1 \in \mathrm{Supp}(S)$ with $\varphi(e_1) = f_1$, which exists in view of (14). By Claim H, every $g \in \mathrm{Supp}(S)$ with $\varphi(g) = f_1$ has $g = e_1$, and if $k_n = 0$ with $\varphi(g_0) = f_1 = \varphi(e_1)$, we can choose $g_0 \in g_0 + \ker \varphi$ appropriately so that $g_0 = e_1$, thereby ensuring that every $g \in \mathrm{Supp}(S^*)$ with $\varphi(g) = f_1$ has $g = e_1$.

Claim I. If $g, h \in \mathrm{Supp}(S)$ with $\varphi(g), \varphi(h) \in \langle f_1 \rangle + f_2$, then $g - h \in \langle e_1 \rangle$.

Proof. Let $S^* = W_0 \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ be a block decomposition with associated index $k_\emptyset = 0$ and associated sequence $S_\sigma = \prod^\bullet_{i \in [1, 2m-2+k_m]} \sigma(W_i)$. Since f_1 is good (by Claim H), we have $\mathsf{v}_{f_1}(\varphi(S)) \le mn - 1$, lest S contain an mn-term zero-sum, contrary to (6). Thus, since each $\varphi(W_i)$, for $i \in [1, 2m-2+k_m]$, either consists of n terms equal to f_1 or no terms equal to f_1 (in view of (8)), it follows that $\mathsf{v}_{f_1}(\varphi(S \cdot W_0^{[-1]})) \le (m-1)n$, meaning there are at least $(2m-2+k_m-(m-1))n \ge mn$ terms of $\varphi(S \cdot W_0^{[-1]})$ from $\langle f_1 \rangle + f_2$. These terms cannot all be equal to each other, lest they would be good by Claim G giving rise to an element with multiplicity at least mn in S, contradicting (6) as before. Therefore

$$|\,\mathrm{Supp}(\varphi(S \cdot W_0^{[-1]})) \setminus \{f_1\}| \ge 2. \tag{17}$$

For $x \in [0, n-1]$, let $L_x \mid \widetilde{W_0}$ be the subsequence of $\widetilde{W_0}$ consisting of all terms g with $\varphi(g) = xf_1 + f_2$, and let $R_x \mid S \cdot W_0^{[-1]}$ be the subsequence of $S \cdot W_0^{[-1]}$ consisting of all terms g with $\varphi(g) = xf_1 + f_2$. Let $I_L \subseteq [0, n-1]$ be all those $x \in [0, n-1]$ with L_x nontrivial, and let $I_R \subseteq [0, n-1]$ be all those $x \in [0, n-1]$ with R_x nontrivial. By a slight abuse of notation, we consider the subscripts on the L_x and R_x modulo n. In view of (17),

$$|I_R| \ge 2.$$

Let $g \in \mathrm{Supp}(\widetilde{W_0})$ and $h \in \mathrm{Supp}(S \cdot W_0^{[-1]})$ be arbitrary with $\varphi(g)$, $\varphi(h) \in \langle f_1 \rangle + f_2$, and let

$$\varphi(g) - \varphi(h) = zf_1 \quad \text{ with } z \in [1, n].$$

Suppose $k_n = 0$ and $\varphi(g_0) \neq f_1$. Then $e_1^{[n-z]} \cdot g \mid \widetilde{W_0}$. Set $W_0' = W_0 \cdot \big(e_1^{[n-z]} \cdot g\big)^{[-1]} \cdot h$, $W_j' = W_j \cdot h^{[-1]} \cdot e_1^{[n-z]} \cdot g$ and $W_i' = W_i$ for $i \neq 0, j$, where $h \in \mathrm{Supp}(W_j)$. Then $S^* = W_0' \cdot W_1' \cdot \ldots \cdot \ldots \cdot W_{2m-2+k_m}$ is a weak block decomposition with $g_0 \in \mathrm{Supp}(W_0')$, associated index

$k'_\emptyset = k_\emptyset = 0$ (as $g \in \mathrm{Supp}(W'_0)$) and associated sequence $S'_\sigma = \prod^\bullet_{i\in[1,2m-2+k_m]} \sigma(W'_i)$. Note that S'_σ is obtained from S_σ by replacing the term $\sigma(W_j)$ by $\sigma(W'_j) = \sigma(W_j) - h + g + (n-z)e_1$. By Lemma 2.1 and Claim B, we must have $S_\sigma = S'_\sigma$, implying $\sigma(W_j) = \sigma(W'_j)$ and $g - h \in \langle e_1\rangle$. As this is true for arbitrary $g \in \mathrm{Supp}(\widetilde{W}_0)$ and $h \in \mathrm{Supp}(S \cdot W_0^{[-1]})$, the claim is complete in this case, allowing us to assume $k_n = 1$ or $\varphi(g_0) = f_1$. In particular, it now follows from (8) that

$$|I_L| \geq 2$$

(as $x_1 + \ldots + x_n \equiv 1 \mod n$ ensures not all x_i are equal to each other).

Suppose $z \geq 2$ for g and h as before. Then $e_1^{[n-z]} \cdot g \mid \widetilde{W}_0$. Set $W'_0 = W_0 \cdot \big(e_1^{[n-z]} \cdot g\big)^{[-1]} \cdot h$, $W'_j = W_j \cdot h^{[-1]} \cdot e_1^{[n-z]} \cdot g$ and $W'_i = W_i$ for $i \neq 0, j$, where $h \in \mathrm{Supp}(W_j)$. Then $S^* = W'_0 \cdot W'_1 \cdot \ldots \cdot W_{2m-2+k_m}$ is a weak block decomposition with associated index $k'_\emptyset$. If $k_n = 0$, then $g_0 \in \mathrm{Supp}(W'_0)$, ensuring $k'_\emptyset = k_\emptyset = 0$; if $k_n = 1$, then $|W'_0| = |W_0| - (n-z) \geq n+1$ follows in view of $z \geq 2$, also ensuring that $k'_\emptyset = k_\emptyset = 0$. Applying Lemma 2.1 and Claim B, we find that g and h are from the same $\langle e_1\rangle$-coset as before.

The argument from the previous paragraph shows that, for any $x \in I_L$, all terms from L_x are from the same $\langle e_1\rangle$-coset as all terms from R_y, for any $y \not\equiv x - 1 \mod n$. If all terms from $\prod^\bullet_{y\in I_R} R_y$ are from the same $\langle e_1\rangle$-coset, then each $x \in I_L$ would have all terms from L_x being from the same $\langle e_1\rangle$-coset as all terms from some R_y with $y \in I_R$ (as $|I_R| \geq 2$), and thus from the same $\langle e_1\rangle$-coset that contains all terms from $\prod^\bullet_{y\in I_R} R_y$. As this would be true for any $x \in I_L$, there would only be one $\langle e_1\rangle$-coset containing all $g \in \mathrm{Supp}(S)$ with $\varphi(g) \in \langle f_1\rangle + f_2$, completing the proof of the claim. Therefore we can instead assume we need at least two $\langle e_1\rangle$-cosets to cover all terms from $\prod^\bullet_{y\in I_R} R_y$. In particular, for any $x \in I_L$, we must have $x - 1 \in I_R$, so $I_L - 1 \subseteq I_R \mod n$. Likewise, since $|I_L| \geq 2$, we can assume we need at least two $\langle e_1\rangle$-cosets to cover all terms from $\prod^\bullet_{x\in I_L} L_x$, and thus for any $y \in I_R$, we have $y + 1 \in I_L$, so $I_R + 1 \subseteq I_L \mod n$. It follows that $|I_L| = |I_R|$ with

$$I_R = \{x - 1 : \ x \in I_L\} \mod n.$$

Suppose $|I_R| \geq 3$. Letting $x_1, x_2 \in I_L$ be distinct, then all terms from $\prod^\bullet_{y\in I_R\setminus\{x_1-1\}} R_y$ are from the same $\langle e_1\rangle$-cost as the terms from L_{x_1}, while all terms from $\prod^\bullet_{y\in I_R\setminus\{x_2-1\}} R_y$ are from the same $\langle e_1\rangle$-cost as the terms from L_{x_2}. Since $|I_R| \geq 3$, there would be a common element $y \in I_R \setminus \{x_1 - 1, x_2 - 1\}$, forcing all terms from $\prod^\bullet_{y\in I_R} R_y$ to be from the same $\langle e_1\rangle$-coset, which we just assumed was not the case. So we instead conclude that $|I_L| = |I_R| = 2$. Let

$$I_L = \{x, y\} \qquad \text{and} \qquad I_R = \{x - 1, y - 1\} \mod n.$$

In view of (8), any W_j with $j \in [1, 2m - 2 + k_m]$ that contains a term h with $\varphi(h) \in \langle f_1\rangle + f_2$ must have all its terms from $\langle f_1\rangle + f_2$. Thus, since $|W_j| = n \geq 3$ and $|I_R| = 2$, the Pigeonhole Principle ensures that there are $h_1 \cdot h_2 \mid W_j$ with $\varphi(h_1) = \varphi(h_2)$, say w.l.o.g. $\varphi(h_1) = \varphi(h_2) = (x-1)f_1 + f_2$. By definition of $I_L = \{x, y\}$, there are $g_1 \cdot g_2 \mid \widetilde{W}_0$ with $\varphi(g_1) = xf_1 + f_2$ and

$\varphi(g_2) = yf_1 + f_2$. Let

$$z' \equiv x + y - 2(x-1) \mod n \quad \text{with } z' \in [1,n]. \tag{18}$$

Suppose $z' \geq 2$. Then $e_1^{[n-z']} \cdot g_1 \cdot g_2 \mid \widetilde{W}_0$. Set $W_0' = W_0 \cdot (e_1^{[n-z']} \cdot g_1 \cdot g_2)^{[-1]} \cdot h_1 \cdot h_2$, $W_j' = W_j \cdot h_1^{[-1]} \cdot h_2^{[-1]} \cdot e_1^{[n-z']} \cdot g_1 \cdot g_2$ and $W_i' = W_i$ for $i \neq 0, j$. Then $S^* = W_0' \cdot W_1' \cdot \ldots \cdot W_{2m-2+k_m}$ is a weak block decomposition say with associated index $k_\emptyset'$ and associated sequence S_σ'. If $k_n = 0$, then $g_0 \in \text{Supp}(W_0')$, ensuring $k_\emptyset' = k_\emptyset = 0$; if $k_n = 1$, then $|W_0'| = |W_0| - (n-z') \geq n+1$ follows in view of $z' \geq 2$, also ensuring that $k_\emptyset' = k_\emptyset = 0$. By Lemma 2.1 and Claim B, it follows that $S_\sigma = S_\sigma'$, and thus

$$\sigma(W_j) = \sigma(W_j') = \sigma(W_j) + (n-z')e_1 + g_1 + g_2 - h_1 - h_2. \tag{19}$$

Since $\varphi(g_2) = yf_1 + f_2$ and $\varphi(h_2) = (x-1)f_2 + f_2$, we have $g_2 \in \text{Supp}(L_y)$ and $h_2 \in \text{Supp}(R_{x-1})$, so our previous argument ensures g_2 and h_2 are from the same $\langle e_1 \rangle$-coset, and then (19) implies that g_1 and h_1 are from the same $\langle e_1 \rangle$-coset. Since $\varphi(g_1) = xf_1 + f_2$ and $\varphi(h_1) = (x-1)f_2 + f_2$, so $g_1 \in \text{Supp}(L_x)$ and $h_1 \in \text{Supp}(R_{x-1})$, the terms from L_x are from the same $\langle e_1 \rangle$-coset as both R_{x-1} and R_{y-1}, contradicting our assumption that we need at least two $\langle e_1 \rangle$-cosets to cover the terms from $\prod^\bullet_{z \in I_R} R_z = R_{x-1} \cdot R_{y-1}$. So we are left to conclude $z' = 1$, which by (18) means

$$y \equiv x - 1 \mod n. \tag{20}$$

Since $I_R = \{x-1, y-1\}$, there is some $j \in [1, 2m-2+k_m]$ and $h \in \text{Supp}(W_j)$ with $\varphi(h) = (y-1)f_1 + f_2$. In view of (8), we have $\text{Supp}(\varphi(W_j)) \subseteq \langle f_1 \rangle + f_2$, and thus all terms from $\varphi(W_j)$ are either equal to $(x-1)f_1 + f_2$ or $(y-1)f_1 + f_2$. Since $\varphi(W_j)$ is an n-term zero-sum, it cannot have a term with multiplicity exactly $n-1$, so there must be $h_1 \cdot h_2 \mid W_j$ with $\varphi(h_1) = \varphi(h_2) = (y-1)f_1 + f_2$. Repeating the argument of the previous paragraph swapping the roles of y and x, we conclude that $x \equiv y - 1 \mod n$. Combined with (20), it follows that $0 \equiv 2 \mod n$, contradicting that $n \geq 3$, which concludes the claim. □

Note that any $g \in \text{Supp}(S)$ with $\varphi(g) \neq f_1$ has $\varphi(g) = xf_1 + f_2$ for some $x \in [0, n-1]$ by Claim C.1. Thus, in view of Claims H and I, we see that $\text{Supp}(S) \subseteq \{e_1\} \cup \big(\langle e_1 \rangle + e_2\big)$ for some $e_2 \in \text{Supp}(S)$. This allow us to apply Lemma 2.2 to S to complete the proof. □

Next, we consider the case when $k_n \in [2, n-1]$.

Proposition 3.2. *Let m, $n \geq 2$ and let $k \in [0, mn-1]$ with $n = k_m n + k_n$, where $k_m \in [0, m-1]$ and $k_n \in [2, n-1]$. Suppose Conjecture 1.1 holds for k_n in $C_n \oplus C_n$. Suppose either Conjecture 1.1 also holds for k_m in $C_m \oplus C_m$, or else $k_m \in [1, m-2]$ and Conjecture 1.1 also holds for $k_m + 1$ in $C_m \oplus C_m$. Then Conjecture 1.1 holds for k in $C_{mn} \oplus C_{mn}$.*

Proof. Let $G = C_{mn} \oplus C_{mn}$ and let $S \in \mathcal{F}(G)$ be a sequence with

$$|S| = 2mn - 2 + k \quad \text{and} \quad 0 \notin \Sigma_{\leq 2mn-1-k}(G). \tag{21}$$

Since $k_m \in [0, m-1]$ and $k_n \in [2, n-1]$, we have $n \geq 3$ and $k = k_m n + k_n \in [2, mn-1]$. Since Conjecture 1.1 is known for $k = mn-1$ (as remarked in the introduction), we can assume $k = k_m n + k_n \in [2, mn-2]$. We need to show Conjecture 1.1.3 holds for S. Let $\varphi : G \to G$ be the multiplication by m homomorphism, so $\varphi(x) = mx$. Note

$$\varphi(G) = mG \cong C_n \oplus C_n \qquad \text{and} \qquad \ker \varphi = nG \cong C_m \oplus C_m.$$

Define a *block decomposition* of S to be a factorization

$$S = W \boldsymbol{\cdot} W_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} W_{2m-2+k_m}$$

with $1 \leq |W_i| \leq n$ and $\varphi(W_i)$ zero-sum for each $i \in [1, 2m-2+k_m]$. Since $\mathsf{s}_{\leq n}(\varphi(G)) = \mathsf{s}_{\leq n}(C_n \oplus C_n) = 3n-2$ and $|S| = (2m-3+k_m)n + 3n - 2 + k_n \geq (2m-3+k_m)n + 3n - 2$, it follows by repeated application of the definition of $\mathsf{s}_{\leq n}(\varphi(G))$ that S has a block decomposition.

Claim A. If $S = W \boldsymbol{\cdot} W_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} W_{2m-2+k_m}$ is a block decomposition of S, then $|W_i| = n$ for all $i \in [1, 2m-2+k_m]$, $|W| = 2n-2+k_n$, $0 \notin \Sigma_{\leq 2n-1-k_n}(\varphi(W))$, and $0 \notin \Sigma_{\leq n-1}(\varphi(S))$. In particular, Conjecture 1.1 holds for $\varphi(W)$.

Proof. Suppose $0 \in \Sigma_{\leq 2n-1-k_n}(\varphi(W))$. Then there is a nontrivial subsequence $W_0 \mid W$ with $|W_0| \leq 2n-1-k_n$ and $\varphi(W_0)$ zero-sum. Now $\sigma(W_0) \boldsymbol{\cdot} \sigma(W_1) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} \sigma(W_{2m-2+k_m})$ is a sequence of $2m-1+k_m$ terms from $\ker \varphi \cong C_m \oplus C_m$. Since $\mathsf{s}_{\leq 2m-1-k_m}(C_m \oplus C_m) = 2m-1+k_m$, it follows that it has a nontrivial zero-sum sequence, say $\prod^\bullet_{i \in I} \sigma(W_i)$ for some nonempty $I \subseteq [0, 2m-2+k_m]$ with $|I| \leq 2m-1-k_m$. But then $\prod^\bullet_{i \in I} W_i$ is a nontrivial zero-sum subsequence of S with

$$|\textstyle\prod^\bullet_{i \in I} W_i| \leq \max\{|W_0|, n\} + (|I|-1)n \leq 2n-1-k_n + (2m-2-k_m)n = 2mn-1-k,$$

contradicting (21). So we instead conclude that $0 \notin \Sigma_{\leq 2n-1-k_n}(\varphi(W))$.

As a result, since $\mathsf{s}_{\leq 2n-1-k_n}(\varphi(G)) = \mathsf{s}_{\leq 2n-1-k_n}(C_n \oplus C_n) = 2n-1+k_n$, and since $|W_i| \leq n$ for all $i \in [1, 2m-2+k_m]$, it follows that

$$2n-2+k_n = 2mn-2+k-(2m-2+k_m)n \leq |S| - \sum_{i=1}^{2m-2+k_m} |W_i| = |W| \leq 2n-2+k_n,$$

forcing equality to hold in our estimates, i.e., $|W_i| = n$ for all $i \in [1, 2m-2+k_m]$ and $|W| = 2n-2+k_n$. If $0 \in \Sigma_{\leq n-1}(\varphi(S))$, then we can find a nontrivial subsequence $W_1' \mid S$ with $\varphi(W_1')$ zero-sum and $|W_1'| \leq n-1$. Applying the argument used to show the existence of a block decomposition, we obtain a block decomposition $S = W' \boldsymbol{\cdot} W_1' \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} W'_{2m-2+k_m}$ with $|W_1'| \leq n-1$, contradicting what was just shown. Therefore $0 \notin \Sigma_{\leq n-1}(\varphi(S))$. Finally, since $|W| = 2n-2+k_n$ and $0 \notin \Sigma_{\leq 2n-1-k_n}(\varphi(W))$ with Conjecture 1.1 holding for k_n in $C_n \oplus C_n$ by hypothesis, it follows that Conjecture 1.1 holds for $\varphi(W)$, completing the claim. $\square$

Suppose

$$S = W \boldsymbol{\cdot} W_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} W_{2m-2+k_m}$$

with each $\varphi(W_i)$ a nontrivial zero-sum for $i \in [1, 2m-2+k_m]$ and $|W| \geq n-1+2k_n$. We call this a *weak block decomposition* of S with associated sequence

$$S_\sigma = \sigma(W_1) \cdot \ldots \cdot \sigma(W_{2m-2+k_m}) \in \mathcal{F}(\ker \varphi).$$

Since $2n-2+k_n \geq n-1+2k_n$, any block decomposition is also a weak block decomposition. Suppose

$$S = W \cdot W_0 \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$$

with each $\varphi(W_i)$ a nontrivial zero-sum for $i \in [0, 2m-2+k_m]$, $|W_i| = n$ for all $i \in [1, 2m-2+k_m]$, and $|W_0| \leq 3n-1-k_n$. We call this an *augmented block decomposition* of S. In such case, $S = (W \cdot W_0) \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ is a block decomposition of S with associated sequence S_σ, and we call

$$\widetilde{S}_\sigma = \sigma(W_0) \cdot S_\sigma = \sigma(W_0) \cdot \sigma(W_1) \cdot \ldots \cdot \sigma(W_{2m-2+k_m}) \in \mathcal{F}(\ker \varphi)$$

the associated sequence for the augmented block decomposition. Conversely, if $S = W \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ is a block decomposition, then Claim A ensures that $|W| = 2n-1+(k_n-1) = \mathsf{s}_{\leq 2n-k_n}(\ker \varphi)$ (in view of $k_n \geq 1$). As a result, there is a nontrivial subsequence $W_0 \mid W$ with $\varphi(W_0)$ zero-sum and $|W_0| \leq 2n-k_n \leq 3n-1-k_n$, ensuring $(W \cdot W_0^{[-1]}) \cdot W_0 \cdot \ldots \cdot W_{2m-2+k_m}$ is an augmented block decomposition, showing such decompositions exist.

Claim B.

1. If $S = W \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ is a weak block decomposition with associated sequence S_σ, then $|S_\sigma| = 2m-2+k_m$ and $0 \notin \Sigma_{\leq 2m-1-k_m}(S_\sigma)$. If it is also a block decomposition, then Conjecture 1.1 holds for S_σ.
2. If $S = W \cdot W_0 \cdot \ldots \cdot W_{2m-2+k_m}$ is an augmented block decomposition with associated sequence $\widetilde{S}_\sigma$, then $|\widetilde{S}_\sigma| = 2m-1+k_m$ and $0 \notin \Sigma_{\leq 2m-2-k_m}(\widetilde{S}_\sigma)$. Moreover, if $k_m \in [0, m-2]$, then Conjecture 1.1 holds for $\widetilde{S}_\sigma$.

Proof. If $S = W \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ is a weak block decomposition with associated sequence S_σ, then $|S_\sigma| = 2m-2+k_m$ holds by definition. If $0 \in \Sigma_{\leq 2m-1-k_m}(S_\sigma)$, then there is some $I \subseteq [1, 2m-2+k_m]$ with $1 \leq |I| \leq 2m-1+k_m$ and $\prod^\bullet_{i\in I} \sigma(W_i)$ zero-sum. In such case, since $|W_i| \geq n$ for all $i \in [1, 2m-2+k_m]$ by Claim A, we find that $\prod^\bullet_{i\in I} W_i$ is a nontrivial zero-sum subsequence of S with length at most

$$\begin{aligned}&(|S|-|W|)-(2m-2+k_m-|I|)n = 2n-2+k_n-|W|+|I|n\\ &\leq 2mn-2+n+k_n-k_mn-|W| \leq 2mn-1-k_mn-k_n = 2mn-1-k,\end{aligned}$$

with the final inequality in view of the definition of a weak block decomposition, which contradicts the hypothesis $0 \notin \Sigma_{\leq 2mn-1-k}(S)$. Therefore $0 \notin \Sigma_{\leq 2m-1-k_m}(S_\sigma)$.

If $S = W \cdot W_0 \cdot \ldots \cdot W_{2m-2+k_m}$ is an augmented block decomposition with associated sequence $\widetilde{S}_\sigma = \sigma(W_0) \cdot S_\sigma$, then $|\widetilde{S}_\sigma| = 2m-1+k_m$ holds by definition. If $0 \in \Sigma_{\leq 2m-2-k_m}(\widetilde{S}_\sigma)$, then there is some $I \subseteq [0, 2m-2+k_m]$ with $1 \leq |I| \leq 2m-2-k_m$ and $\prod^\bullet_{i\in I} \sigma(W_i)$ zero-sum.

In such case, since $|W_i| = n$ for all $i \geq 1$ and $|W_0| \leq 3n - 1 - k_n$, we find that $\prod^\bullet_{i\in I} W_i$ is a nontrivial zero-sum subsequence of S with length at most $\max\{|I|n, |W_0| + (|I| - 1)n\} \leq 3n-1-k_n+(2m-3-k_m)n = 2mn-1-k$, which contradicts the hypothesis $0 \notin \Sigma_{\leq 2mn-1-k}(S)$. Therefore $0 \notin \Sigma_{\leq 2m-2-k_m}(\widetilde{S}_\sigma)$.

Suppose $S = W \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ is a block decomposition with associated sequence S_σ. As noted above Claim B, there exists some $W_0 \mid W$ such that $(W \cdot W_0^{[-1]}) \cdot W_0 \cdot \ldots \cdot W_{2m-2+k_m}$ is an augmented block decomposition with associated sequence $\sigma(W_0) \cdot S_\sigma$. As already shown, $0 \notin \Sigma_{\leq 2m-2-k_m}(\sigma(W_0) \cdot S_\sigma)$ with $|\sigma(W_0) \cdot S_\sigma| = 2m - 1 + k_m$. By hypothesis, either Conjecture 1.1 holds for k_m in $C_m \oplus C_m$, or else $k_m \in [1, m - 2]$ and Conjecture 1.1 holds for $k_m + 1$ in $C_m \oplus C_m$. In the former case, Conjecture 1.1 holds for S_σ in view of the already established $|S_\sigma| = 2m-2+k_m$ and $0 \notin \Sigma_{\leq 2m-1-k_m}(S_\sigma)$. In the latter case, Conjecture 1.1 holds for $\sigma(W_0)\cdot S_\sigma$ in view of the already established $|\sigma(W_0) \cdot S_\sigma| = 2m - 1 + k_m$ and $0 \notin \Sigma_{\leq 2m-2-k_m}(\Sigma(W_0) \cdot S_\sigma)$, which combined with Lemma 2.4 ensures that it does so with respect to some basis (f_1, f_2) with $\sigma(W_0) = f_1 + f_2$ and Conjecture 1.1 holding for S_σ. Thus Conjecture 1.1 holds for S_σ in both cases.

Suppose $S = W \cdot W_0 \cdot \ldots \cdot W_{2m-2+k_m}$ is an augmented block decomposition with associated sequence $\widetilde{S}_\sigma = \sigma(W_0) \cdot S_\sigma$. As noted above Claim B, $(W \cdot W_0) \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ is a block decomposition of S with associated sequence S_σ. As already shown, $|S_\sigma| = 2m - 2 + k_m$ and $0 \notin \Sigma_{\leq 2m-1-k_m}(S_\sigma)$ with Conjecture 1.1 holding for S_σ. As a result, if $k_m \in [1, m - 2]$, then Lemma 2.3 implies that Conjecture 1.1 holds for $\widetilde{S}_\sigma$. On the other hand, If $k_m = 0$, then Conjecture 1.1 is known to hold without condition for k_m and $k_m + 1$ in $C_m \oplus C_m$, ensuring that Conjecture 1.1 holds for $\widetilde{S}_\sigma$. Thus Conjecture 1.1 holds for $\widetilde{S}_\sigma$ in both cases, completing the claim. □

Claim C. There exists a basis $(\overline{e}_1, \overline{e}_2)$ for $\varphi(G)$ such that $\mathrm{Supp}(\varphi(S)) = \{\overline{e}_1, \overline{e}_2, x\overline{e}_1 + \overline{e}_2\}$ for some $x \in [1, n - 1]$ with $\gcd(x, n) = 1$ and either $x = 1$ or $k_n = n - 1$. In particular, any block decomposition $S = W \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ has $\varphi(W) = \overline{e}_1^{[n-1]} \cdot \overline{e}_2^{[n-1]} \cdot (x\overline{e}_1 + \overline{e}_2)^{[k_n]}$ with $\varphi(W_i) \in \{\overline{e}_1^{[n]}, \overline{e}_2^{[n]}, (x\overline{e}_1 + \overline{e}_2)^{[n]}\}$ for all $i \in [1, 2m - 2 + k_m]$.

Proof. Let $S = W \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ be an arbitrary block decomposition. Then Claim A ensures that Conjecture 1.1 holds for $\varphi(W)$, so there exists a basis $(\overline{e}_1, \overline{e}_2)$ for $\varphi(G)$ such that

$$\varphi(W) = \overline{e}_1^{[n-1]} \cdot \overline{e}_2^{[n-1]} \cdot (x\overline{e}_1 + \overline{e}_2)^{[k_n]} \tag{22}$$

for some $x \in [1, n - 1]$ with $\gcd(x, n) = 1$ and either $x = 1$ or $k_n = n - 1$ (as we have $k_n \in [2, n - 1]$). Since $(\overline{e}_1, \overline{e}_2)$ is a basis and $\gcd(x, n) = 1$, any n-term zero-sum sequence with support contained in $\{\overline{e}_1, \overline{e}_2, x\overline{e}_1 + \overline{e}_2\}$ must have support of size 1. Consequently, if we can show that $|\mathrm{Supp}(\varphi(S))| = 3$, then any block decomposition $S = W' \cdot W'_1 \cdot \ldots \cdot W'_{2m-2+k_m}$ will have $\varphi(W'_i) \in \{\overline{e}_1^{[n]}, \overline{e}_2^{[n]}, (x\overline{e}_1 + \overline{e}_2)^{[n]}\}$ for all $i \in [1, 2m - 2 + k_m]$. Moreover, if $k_n = n - 1$, then Conjecture 1.1 must hold for $\varphi(W')$ with $\mathrm{Supp}(\varphi(W')) \subseteq \{\overline{e}_1, \overline{e}_2, x\overline{e}_1 + \overline{e}_2\}$, with each term having multiplicity $n-1$, which forces $\varphi(W') = \varphi(W) = \overline{e}_1^{[n-1]} \cdot \overline{e}_2^{[n-1]} \cdot (x\overline{e}_1 + \overline{e}_2)^{[n-1]}$, while for

$k_n \le n-2$ and $x=1$, Conjecture 1.1 must hold for $\varphi(W')$ with $\mathrm{Supp}(\varphi(W')) \subseteq \{\overline{e}_1, \overline{e}_2, \overline{e}_1+\overline{e}_2\}$, $\overline{e}_1 + (\overline{e}_1+\overline{e}_2) \ne \overline{e}_2$ and $\overline{e}_2 + (\overline{e}_1+\overline{e}_2) \ne \overline{e}_1$ (as $n \ge 3$), ensuring that $\varphi(W') = \varphi(W) = \overline{e}_1^{[n-1]} \cdot \overline{e}_2^{[n-1]} \cdot (x\overline{e}_1+\overline{e}_2)^{[k_n]}$. In both cases, the claim would be complete. Thus it suffices to show $|\mathrm{Supp}(\varphi(S))| = 3$. Assume by contradiction that $|\mathrm{Supp}(\varphi(S))| > 3$, meaning there is some $g \in \mathrm{Supp}(S \cdot W_0^{[-1]})$ with $\varphi(g) \notin \{\overline{e}_1, \overline{e}_2, x\overline{e}_1+\overline{e}_2\} = \mathrm{Supp}(\varphi(W))$, and w.l.o.g. $g \in \mathrm{Supp}(W_1)$.

Suppose there were two distinct elements from $\mathrm{Supp}(\varphi(W \cdot W_1))$ each with multiplicity at least n in $\varphi(W \cdot W_1)$. Then it would be possible to re-factorize $W \cdot W_1 = W' \cdot W_1'$ with $|W_1'| = |W_1| = n$, with $\varphi(W_1')$ a zero-sum sequence having support of size 1, and with W' containing a zero-sum subsequence of length n and support 1. But then $S = W' \cdot W_1' \cdot W_2 \cdot \ldots \cdot W_{2m-2+k_m}$ would be a block decomposition with $0 \in \Sigma_{\le n}(\varphi(W'))$, contrary to Claim A. So we conclude there can be at most one term from $\varphi(W \cdot W_1)$ with multiplicity at least n.

Suppose $\mathsf{v}_{\varphi(g)}(\varphi(W \cdot W_1)) \ge n-1$. Since $\varphi(g) \notin \mathrm{Supp}(\varphi(W))$, this ensures $\varphi(g)$ has multiplicity at least $n-1$ in the n-term zero-sum sequence $\varphi(W_1)$, which is only possible if $\varphi(W_1) = \varphi(g)^{[n]}$. In such case, all terms in $\varphi(W \cdot W_1 \cdot g^{[-1]})$ have multiplicity at most $n-1$. Since we have $|W \cdot W_1 \cdot g^{[-1]}| = 3n-3+k_n \ge 3n-1 \ge \mathsf{s}_{\le n}(\varphi(G))$, it follows that there is a nontrivial subsequence $W_1' \mid W \cdot W_1 \cdot g^{[-1]}$ with $\varphi(W_1')$ a zero-sum sequence of length at most n. Setting $W' = W \cdot W_1 \cdot (W_1')^{[-1]}$, it follows that $S = W' \cdot W_1' \cdot W_2 \cdot \ldots \cdot W_{2m-2+k_m}$ is a block decomposition with $|\mathrm{Supp}(\varphi(W' \cdot W_1'))| = |\mathrm{Supp}(\varphi(W \cdot W_1))| = 4$ and $|\mathrm{Supp}(\varphi(W_1'))| > 1$ (as $W_1' \mid W \cdot W_1 \cdot g^{[-1]}$ with all terms of $\varphi(W \cdot W_1 \cdot g^{[-1]})$ having multiplicity at most $n-1$). Replacing the initial block decomposition by $S = W' \cdot W_1' \cdot W_2 \cdot \ldots \cdot W'_{2m-2+k_m}$ and repeating all arguments from the start (including possibly redefining the elements $\overline{e}_1$, $\overline{e}_2$ and $x\overline{e}_1+\overline{e}_2$), we see that we can w.l.o.g. assume there is some $g \in \mathrm{Supp}(W_1)$ with

$$\varphi(g) \notin \mathrm{Supp}(\varphi(W)) = \{\overline{e}_1, \overline{e}_2, x\overline{e}_1+\overline{e}_2\} \quad \text{ and } \quad \mathsf{v}_{\varphi(g)}(\varphi(W \cdot W_1)) \le n-2.$$

Let $g_1, g_2 \in \mathrm{Supp}(W)$ be elements with $\varphi(g_1) = \overline{e}_1$ and $\varphi(g_2) = \overline{e}_2$. Now we see that $W \cdot W_1 \cdot g_1^{[-1]} \cdot g_2^{[-1]} \cdot g^{[-1]}$ is a sequence of length $3n-5+k_n \ge 3n-3$. As a result, if $\varphi(W \cdot W_1 \cdot g_1^{[-1]} \cdot g_2^{[-1]} \cdot g^{[-1]})$ does not contain a zero-sum subsequence of length at most n, then the established case of Conjecture 1.1.4 implies that $\mathrm{Supp}(\varphi(W \cdot W_1 \cdot g_1^{[-1]} \cdot g_2^{[-1]} \cdot g^{[-1]})) = 3$ with each of these three terms occurring with multiplicity $n-1$ in $\varphi(W \cdot W_1 \cdot g_1^{[-1]} \cdot g_2^{[-1]} \cdot g^{[-1]})$. However, since $n \ge 3$, (22) ensures that $\overline{e}_1, \overline{e}_2 \in \mathrm{Supp}(\varphi(W \cdot W_1 \cdot g_1^{[-1]} \cdot g_2^{[-1]} \cdot g^{[-1]}))$, meaning $\overline{e}_1$ and $\overline{e}_2$ each have multiplicity $n-1$ in $\varphi(W \cdot W_1 \cdot g_1^{[-1]} \cdot g_2^{[-1]} \cdot g^{[-1]})$, whence $\mathsf{v}_{\overline{e}_1}(\varphi(W \cdot W_1)) \ge n$ and $\mathsf{v}_{\overline{e}_2}(\varphi(W \cdot W_1)) \ge n$ (since $\varphi(g_1) = \overline{e}_1$ and $\varphi(g_2) = \overline{e}_2$), contradicting that we showed earlier that at most one term of $\varphi(W \cdot W_1)$ can have multiplicity at least n. Therefore we instead conclude that there is some subsequence $W_1' \mid W \cdot W_1 \cdot g_1^{[-1]} \cdot g_2^{[-1]} \cdot g^{[-1]}$ with $\varphi(W_1')$ a length n zero-sum (in view of Claim A). Setting $W' = W \cdot W_1 \cdot (W_1')^{[-1]}$, we find that $S = W' \cdot W_1' \cdot W_2 \cdot \ldots \cdot W_{2m-2+k_m}$ is a block decomposition with $g, g_1, g_2 \in \mathrm{Supp}(W')$ and $\varphi(g_1) = \overline{e}_1$, $\varphi(g_2) = \overline{e}_2$ and $\varphi(g) \notin \{\overline{e}_1, \overline{e}_2, x\overline{e}_1+\overline{e}_2\}$.

Applying our initial argument in Claim C using this new block decomposition, we immediately obtain a contradiction unless Conjecture 1.1 holds for $\varphi(W')$ with $\mathrm{Supp}(\varphi(W')) = \{\overline{e}_1, \overline{e}_2, \varphi(g)\}$. If $k_n = n-1$, this forces each of the terms $\overline{e}_1$, $\overline{e}_2$ and $\varphi(g)$ to have multiplicity $n-1$ in $\varphi(W')$, contradicting that $W' \mid W \cdot W_1$ with the multiplicity of $\varphi(g)$ in $\varphi(W \cdot W_1)$ at most $n-2$ (as shown above). Therefore we must have $k_n \le n-2$, in which case $x = 1$, and then, as $\varphi(g)$ has multiplicity at most $n-2$, the only way Conjecture 1.1 can hold for $\varphi(W')$ is if it does so with basis $(\overline{e}_1, \overline{e}_2)$ and $\varphi(g) = \overline{e}_1 + \overline{e}_2$, contradicting that $\varphi(g) \notin \{\overline{e}_1, \overline{e}_2, x\overline{e}_1 + \overline{e}_2\} = \{\overline{e}_1, \overline{e}_2, \overline{e}_1 + \overline{e}_2\}$. This completes Claim C. □

We define a term $g \in \mathrm{Supp}(S)$ to be *good* if $g, h \in \mathrm{Supp}(S)$ with $\varphi(g) = \varphi(h)$ implies $g = h$. A term $g \in \mathrm{Supp}(\varphi(S))$ is *good* if $\mathrm{Supp}(S)$ contains exactly one element from $\varphi^{-1}(g)$. Then, for $g \in \mathrm{Supp}(S)$, we find that $\varphi(g)$ is good if and only if g is good.

Let $(\overline{e}_1, \overline{e}_2)$ be an arbitrary basis for $\ker\varphi$ for which Claim C holds with

$$\mathrm{Supp}(\varphi(S)) = \{\overline{e}_1, \overline{e}_2, x\overline{e}_1 + \overline{e}_2\}, \tag{23}$$

where $x \in [1, n-1]$ with $\gcd(x, n) = 1$ and either $x = 1$ or $k_n = n-1$. Let $e_1, e_2 \in G$ be, for the moment, arbitrary representatives for $\overline{e}_1$ and $\overline{e}_2$, so $\varphi(e_1) = \overline{e}_1$, $\varphi(e_2) = \overline{e}_2$ and $\varphi(xe_1 + e_2) = x\overline{e}_1 + \overline{e}_2$. We divide the remainder of the proof into two main cases. We remark that the cases $k_m \in [1, m-2]$ could be handled by the methods of either CASE 1 or 2, but there is some simplification to the arguments by including them in CASE 2.

CASE 1: $k_m = m-1$.

We begin with the following claim.

Claim D.1. All terms $g \in \mathrm{Supp}(S)$ are good.

Proof. Let $S = W \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ be an arbitrary block decomposition. In view of Claim C, let $g_1, g_2, g_3 \in \mathrm{Supp}(W)$ be arbitrary with $\varphi(g_1) = \overline{e}_1$, $\varphi(g_2) = \overline{e}_2$ and $\varphi(g_3) = x\overline{e}_1 + \overline{e}_2$. Let $S_\sigma = \sigma(W_1) \cdot \ldots \cdot \sigma(W_{2m-2+k_m})$ be the associated sequence, which satisfies Conjecture 1.1 by Claim B. If $h \in \mathrm{Supp}(S \cdot W^{[-1]})$, say $h \in \mathrm{Supp}(W_j)$, then $\varphi(h) = \varphi(g_k)$ for some $k \in [1,3]$ by Claim C. Setting $W' = W \cdot g_k^{[-1]} \cdot h$, $W'_j = W_j \cdot h^{[-1]} \cdot g_k$ and $W'_i = W_i$ for all $i \neq j$, we obtain a new block decomposition $S = W' \cdot W'_1 \cdot \ldots \cdot W'_{2m-2+k_m}$ with associated sequence $S'_\sigma = \sigma(W'_1) \cdot \ldots \cdot \sigma(W'_{2m-2+k_m})$ also satisfying Conjecture 1.1 by Claim B. Since $k_m = m-1 \ge 1$, we can then apply Lemma 2.1 to conclude that $S'_\sigma = S_\sigma$ and $\sigma(W'_j) = \sigma(W_j)$. Since $\sigma(W'_j) = \sigma(W_j) - h + g_k$, this forces $g_k = h$. Ranging over all $h \in \mathrm{Supp}(S \cdot W^{[-1]})$ and $g_k \in \mathrm{Supp}(W)$ with $\varphi(g_k) = \varphi(h)$ now shows that $\varphi(h)$ is good.

In summary, this argument shows that all terms occurring in $\varphi(S \cdot W^{[-1]})$ are good. Consequently, since Claim C ensures that $|\mathrm{Supp}(\varphi(S))| = 3$, the only way Claim D can fail is if $|\mathrm{Supp}(\varphi(S \cdot W^{[-1]}))| \le 2$ with all terms in $\mathrm{Supp}(\varphi(S \cdot W^{[-1]}))$ good. In view of Claim C, each $\varphi(W_i)$ with $i \in [1, 2m-2+k_m]$ consists of a single term from $\{\overline{e}_1, \overline{e}_2, x\overline{e}_1 + \overline{e}_2\}$ repeated n times. As a result, if $|\mathrm{Supp}(\varphi(S \cdot W^{[-1]}))| \le 2$, then the Pigeonhole Principle ensures that, among the terms from the $2m-2+k_m = 3m-3$ blocks W_i with $i \in [1, 2m-2+k_m]$, at least $\lceil \frac{3m-3}{2} \rceil \geq m$ of these blocks $\varphi(W_i)$ must have support equal to the same element, which is then good, ensuring that there is only a single distinct term among these blocks W_i. In such case, S has a term with multiplicity at least mn, contradicting that $0 \notin \Sigma_{\leq 2mn-1-k}(S)$. Therefore, we instead conclude that $|\operatorname{Supp}(\varphi(S \cdot W^{[-1]}))| = 3$, meaning all $g \in \operatorname{Supp}(S)$ are good as explained above. □

Since $k_m = m-1$ and $k = k_m n + k_n \in [2, mn-2]$, we have $k_n \neq n-1$, meaning $k_n \in [2, n-2]$ with $n \geq 4$. This ensures that $x = 1$ in Claim C. In view of Claim D.1,

$$\operatorname{Supp}(S) = \{e_1, e_2, e_1 + e_2 + \alpha\} \tag{24}$$

for some $e_1, e_2 \in G$ and $\alpha \in \ker \varphi$ with $\varphi(e_1) = \overline{e}_1$, $\varphi(e_2) = \overline{e}_2$ and $\varphi(e_1 + e_2 + \alpha) = \overline{e}_1 + \overline{e}_2$. Let $S = W \boldsymbol{\cdot} W_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} W_{2m-2+k_m}$ be an arbitrary block decomposition. Then Claim C implies that

$$W = e_1^{[n-1]} \boldsymbol{\cdot} e_2^{[n-1]} \boldsymbol{\cdot} (e_1 + e_2 + \alpha)^{[k_n]}$$

with $k_n \in [2, n-2]$. Moreover, we can partition $[1, 2m-2+k_m] = I_1 \cup I_2 \cup I_3$ with I_j consisting of all indices $i \in [1, 2m-2+k_m]$ such that $W_i = e_j^{[n]}$ (for $j \in [0,1]$) or such that $W_i = (e_1+e_2+\alpha)^{[n]}$ (for $j = 3$). If $I_j = \emptyset$ for some $j \in [1,3]$, then, since $2m-2+k_m \geq 2m-1$ (as $k_m = m-1 \geq 1$), the Pigeonhole Principle ensures that $|I_{j'}| \geq m$ for some $j' \in [1,3] \setminus \{j\}$. In such case, S has a term with multiplicity at least mn, contradicting that $0 \notin \Sigma_{\leq mn}(S)$ by hypothesis. Therefore we may assume each I_j for $j \in [1,3]$ is nonempty.

Since $I_3 \neq \emptyset$, let $j \in I_3$. Set $W' = W \boldsymbol{\cdot} e_1^{[-1]} \boldsymbol{\cdot} e_2^{[-1]} \boldsymbol{\cdot} (e_1+e_2+\alpha)$, $W'_j = W_j \boldsymbol{\cdot} (e_1+e_2+\alpha)^{[-1]} \boldsymbol{\cdot} e_1 \boldsymbol{\cdot} e_2$, and $W'_i = W_i$ for all $i \neq j$. Since $|W'| = |W| - 1 = 2n-3+k_n \geq n-1+2k_n$ (in view of $k_n \in [2, n-2]$), we see that $S = W' \boldsymbol{\cdot} W'_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} W'_{2m-2+k_m}$ is a weak block decomposition with associated sequence $S'_\sigma = \sigma(W'_1) \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} \sigma(W'_{2m-2+k_m})$. Since Conjecture 1.1 is known to always hold for $k_m = m-1$, it follows in view of Claim B that Conjecture 1.1 holds for S'_σ. As a result, since the sequence S'_σ is obtained from S_σ by replacing the term $\sigma(W_j)$ by $\sigma(W'_j) = \sigma(W_j) - \alpha$, and since Conjecture 1.1 holds for S_σ by Claim B, we can apply Lemma 2.1 (as $k_m = m-1 \geq 1$) to conclude $\alpha = 0$. But now $\operatorname{Supp}(S) \subseteq \{e_1, e_2, e_1 + e_2\}$ by (24), so applying Lemma 2.2 yields the desired structure for S, completing CASE 1.

CASE 2: $k_m \in [0, m-2]$.

Let $S = W \boldsymbol{\cdot} W_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} W_{2m-2+k_m}$ be an arbitrary block decomposition with associated sequence S_σ. Then Claim C implies that

$$\varphi(W) = \overline{e}_1^{[n-1]} \boldsymbol{\cdot} \overline{e}_2^{[n-1]} \boldsymbol{\cdot} (x\overline{e}_1 + \overline{e}_2)^{[k_n]} \tag{25}$$

with $k_n \in [2, n-1]$, with $x \in [1, n-1]$ and $\gcd(x, n) = 1$, and with either $x = 1$ or $k_n = n-1$. Since $n \geq 3$ (as noted at the start of the proof), $x = n-1$ is only possible if $k_n = n-1$, in which case Claim C also holds replacing the basis (f_1, f_2) by $(f_1, xf_1 + f_2) = (f_1, -f_1 + f_2)$ with $f_2 = f_1 + (xf_1 + f_2)$. In such case, by using this alternative basis in Claim C, we obtain $x = 1 < n-1$. This allows us to w.l.o.g. assume $x \in [1, n-2]$ with $\gcd(x, n) = 1$.

If $x = 1$, then (25) and $k_n \in [2, n-1]$ ensures that there are sequences $U_0 \mid W$ and $V_0 \mid W$ with

$$\varphi(U_0) = \overline{e}_1^{[n-k_n]} \cdot \overline{e}_2^{[n-k_n]} \cdot (\overline{e}_1 + \overline{e}_2)^{[k_n]} \quad \text{and} \quad \varphi(V_0) = \overline{e}_1^{[n-k_n+1]} \cdot \overline{e}_2^{[n-k_n+1]} \cdot (\overline{e}_1 + \overline{e}_2)^{[k_n-1]}.$$

Since $k_n \in [2, n-1]$, we find that

$$\text{Supp}(\varphi(U_0)) = \text{Supp}(\varphi(V_0)) = \{\overline{e}_1, \overline{e}_2, \overline{e}_1 + \overline{e}_2\}, \tag{26}$$

$$\overline{e}_1, \overline{e}_2 \in \text{Supp}(\varphi(W \cdot U_0^{[-1]})) \quad \text{and} \quad \overline{e}_1 + \overline{e}_2 \in \text{Supp}(\varphi(W \cdot V_0^{[-1]})). \tag{27}$$

On the other hand, if $x \neq 1$, then $x \in [2, n-2]$ with $\gcd(x, n) = 1$, $n \geq 5$ and $k_n = n-1$. In such case, let $x^* \in [2, n-2]$ be the multiplicative inverse of $-x$, so

$$xx^* \equiv -1 \mod n.$$

In this case, in view of (25) and $n \geq 5$, there is a sequence $W_0 \mid W$ with

$$\varphi(W_0) = \overline{e}_1 \cdot \overline{e}_2^{[n-x^*]} \cdot (x\overline{e}_1 + \overline{e}_2)^{[x^*]}.$$

In view of $x^* \in [2, n-2]$ and $k_n = n-1$, we have

$$\text{Supp}(\varphi(W_0)) = \text{Supp}(\varphi(W \cdot W_0^{[-1]})) = \{\overline{e}_1, \overline{e}_2, x\overline{e}_1 + \overline{e}_2\} \quad \text{and} \tag{28}$$

$$\mathsf{v}_{\overline{e}_1}(\varphi(W \cdot W_0^{[-1]})) = n - 2 \geq x. \tag{29}$$

Since $|W_0| = n+1$, $|U_0| = 2n - k_n$ and $|V_0| = 2n - k_n + 1$ are all at most $3n - 1 - k_n$ (in view of $n \geq 2$ and $k_n \leq n - 1 \leq 2n - 2$), it follows that $S = (W \cdot W_0^{[-1]}) \cdot W_0 \cdot W_1 \ldots \cdot W_{2m-2+k_m}$, $S = (W \cdot U_0^{[-1]}) \cdot U_0 \cdot W_1 \ldots \cdot W_{2m-2+k_m}$ and $S = (W \cdot V_0^{[-1]}) \cdot V_0 \cdot W_1 \ldots \cdot W_{2m-2+k_m}$ are each augmented block decompositions of S (when they are defined), with respective associated sequences $\sigma(W_0) \cdot S_\sigma$, $\sigma(U_0) \cdot S_\sigma$, and $\sigma(V_0) \cdot S_\sigma$. In view of Claim B and the case hypothesis $k_m \in [0, m-2]$, Conjecture 1.1 holds for all of these associated sequences.

We continue with the following claim.

Claim D.2. All terms $g \in \text{Supp}(S)$ are good.

Proof. Suppose $x \neq 1$. Consider the augmented block decomposition $S = (W \cdot W_0^{[-1]}) \cdot W_0 \cdot W_1 \ldots \cdot W_{2m-2+k_m}$. Then (28) ensures that there are $g_1, g_2, g_2 \in \text{Supp}(W \cdot W_0^{[-1]})$ with $\varphi(g_1) = \overline{e}_1$, $\varphi(g_2) = \overline{e}_2$ and $\varphi(g_3) = x\overline{e}_1 + \overline{e}_2$. Taking an arbitrary $g \in \text{Supp}(W_0 \cdot \ldots \cdot W_{2m-2+k_m})$ and exchanging g with g_j, where g_j with $j \in [1, 3]$ is the element with $\varphi(g_j) = \varphi(g)$, results in a new augmented block decomposition. Applying Lemma 2.1, we find that the associated sequence for the new block decomposition must equal that of the original one, forcing $g = g_j$. Ranging over all possible $g_j \in \text{Supp}(W \cdot W_0^{[-1]})$ and $g \in \text{Supp}(W_0 \cdot \ldots \cdot W_{2m-2+k_m})$ with $\varphi(g) = \varphi(g_j)$, it follows that g is good. This shows that all terms from $W_0 \cdot \ldots \cdot W_{2m-2+k_m}$ are good, and in view of (28), each possible term $\overline{e}_1$, $\overline{e}_2$ and $x\overline{e}_1 + \overline{e}_2$ occurs in $\varphi(W_0 \cdot \ldots \cdot W_{2m-2+k_m})$, ensuring that every $g \in \text{Supp}(S)$ is good.

Next suppose $x = 1$. Repeating the above argument using the augmented block decomposition $S = (W \cdot U_0^{[-1]}) \cdot U_0 \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ in place of $S = (W \cdot W_0^{[-1]}) \cdot W_0 \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$, and using (27) and (26) in place of (28) shows that $\overline{e}_1$ and $\overline{e}_2$ are both good. Repeating the above argument using the augmented block decomposition $S = (W \cdot V_0^{[-1]}) \cdot V_0 \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ in place of $S = (W \cdot W_0^{[-1]}) \cdot W_0 \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$, and using (27) and (26) in place of (28) shows that $\overline{e}_1 + \overline{e}_2$ is good. As $\text{Supp}(\varphi(S)) = \{\overline{e}_1, \overline{e}_2, \overline{e}_1 + \overline{e}_2\}$, this ensures that all terms $g \in \text{Supp}(S)$ are good. □

In view of Claim D.2, there are $e_1, e_2 \in G$ and $\alpha \in \ker \varphi$ such that

$$\text{Supp}(S) = \{e_1, e_2, xe_1 + e_2 + \alpha\}$$

with $\varphi(e_1) = \overline{e}_1$, $\varphi(e_2) = \overline{e}_2$ and $\varphi(e_1 + e_2 + \alpha) = x\overline{e}_1 + \overline{e}_2$.

Suppose $x \neq 1$. Then (28) and (29) ensure that there is a subsequence $T \mid W \cdot W_0^{[-1]}$ with $T = e_1^{[x]} \cdot e_2$. In view of (28), we have $xe_2 + e_3 + \alpha \in \text{Supp}(W_0)$, so set $W_0' = W_0 \cdot (xe_1 + e_2 + \alpha)^{[-1]} \cdot T$. Since $|W_0'| = |W_0| + x = n + 1 + x \leq 2n = 3n - 1 - k_n$, it follows that $(W \cdot (W_0')^{[-1]}) \cdot W_0' \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ is an augmented block decomposition with associated sequence satisfying Conjecture 1.1 by Claim B. Applying Lemma 2.1, we find that the associated sequence for the new block decomposition must equal that of the original one, which is only possible if $\alpha = 0$. We now have $\text{Supp}(S) = \{e_1, e_2, xe_1 + e_2 + \alpha\} = \{e_1, e_2, xe_1 + e_2\}$, so applying Lemma 2.2 yields the desired structure for S.

Next Suppose $x = 1$. Then (27) ensures that there is a subsequence $T \mid W \cdot U_0^{[-1]}$ with $T = e_1 \cdot e_2$. In view of (26), we have $e_1 + e_2 + \alpha \in \text{Supp}(U_0)$, so set $U_0' = U_0 \cdot (e_1 + e_2 + \alpha)^{[-1]} \cdot T$. Since $|U_0'| = |U_0| + 1 = 2n - k_n + 1 \leq 3n - 1 - k_n$, it follows that $(W \cdot (U_0')^{[-1]}) \cdot U_0' \cdot W_1 \cdot \ldots \cdot W_{2m-2+k_m}$ is an augmented block decomposition with associated sequence satisfying Conjecture 1.1 by Claim B. Applying Lemma 2.1, we find that the associated sequence for the new block decomposition must equal that of the original one, which is only possible if $\alpha = 0$. As before, we now have $\text{Supp}(S) = \{e_1, e_2, e_1 + e_2 + \alpha\} = \{e_1, e_2, e_1 + e_2\}$, so applying Lemma 2.2 yields the desired structure for S, which completes CASE 2 and the proof. □

Proof of Theorem 1.2. Theorem 1.2 follows directly from Propositions 3.1 and 3.2 □

We conclude the paper by giving the short proofs of Corollaries 1.3, 1.4 and 1.5.

Proof of Corollary 1.3. It suffices in view of Theorem 1.2 to know Conjecture 1.1 holds for all $k \in [0, p-1]$ in $G = C_p \oplus C_p$, for $p \in \{2, 3, 5, 7\}$. As noted in the introduction, Conjecture 1.1 is known for $k \leq 1$, $k = p - 1$ and $k \in [2, \frac{2p+1}{3}]$ in $C_p \oplus C_p$ with p prime. Since, $p - 2 \leq \frac{2p+1}{3}$ for $p \leq 7$, this means Conjecture 1.1 is known for all $k \in [0, p-1]$ in $C_p \oplus C_p$ for $p \leq 7$ prime, as required. □

Proof of Corollary 1.4. In view of Corollary 1.3, we can assume $n \geq 11$. As noted in the introduction, Conjecture 1.1 is known for $k \leq \frac{2n+1}{3}$ in a p-group $C_n \oplus C_n$ provided $p \nmid k$ and $n \geq 5$. It remains to show Conjecture 1.1 holds for $k = rp$ with $r \in [1, \frac{2n+1}{3p}]$. If $n = p$, there is nothing to show, so we assume $n = p^s$ with $s \geq 2$, and proceed by induction on s with the base $s = 1$ of the induction complete. We have $rp = k \leq \frac{2p^s+1}{3}$, ensuring that $r \leq \frac{2p^{s-1}+1}{3}$. Thus, by induction hypothesis, Conjecture 1.1 holds for r in $C_{p^{s-1}} \oplus C_{p^{s-1}}$, while Conjecture 1.1 holds in general for $k_p := 0$ in $C_p \oplus C_p$ (as noted in the introduction). As a result, Theorem 1.2 (applied with $n = p$ and $m = p^{s-1}$) implies that Conjecture 1.1 holds for $k = rp$ in $C_{p^s} \oplus C_{p^s}$, completing the induction and the proof. □

Proof of Corollary 1.5. Write $n = bd$ with b and d proper nontrivial divisors of n. As noted in the introduction, Conjecture 1.1 holds for $k = d - 1$ and for $k = 1$ in $C_d \oplus C_d$; if $b = 2$, then Conjecture 1.1 holds for $k = b - 2 = 0$ in $C_b \oplus C_b$; and if $b \geq 3$, then Conjecture 1.1 holds for $k + 1 = b - 1$ in $C_b \oplus C_b$. In both of the latter two cases, since $d - 1 \geq 1$, applying Theorem 1.2 using $n = d$ and $m = b$ shows that Conjecture 1.1 holds for $k = n - d - 1 = (b - 2)d + (d - 1)$ in $C_{bd} \oplus C_{bd} = C_n \oplus C_n$, and applying Theorem 1.2 using $n = d$ and $m = b$ shows that Conjecture 1.1 holds for $k = n - 2d + 1 = (b - 2)d + 1$ in $C_{bd} \oplus C_{bd} = C_n \oplus C_n$, as desired. □

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