PLANAR SUBSETS WITH SMALL DOUBLING

DAVID J. GRYNKIEWICZ

ABSTRACT. Let A, $B \subseteq \mathbb{Z}^2$ be finite, nonempty subsets each covered by precisely 2 horizontal lines. Suppose $\langle A+B-A-B\rangle = \mathbb{Z}^2$, $|A| \geq |B|$ and $|A+B| = |A|+2|B|-3+r \leq |A|+\frac{19}{7}|B|-5$. Then there exist subsets P_A, P_B, $P \subseteq \mathbb{Z}^2$, each the union of two arithmetic progressions with difference $(1,0)$, such that $A \subseteq P_A$, $B \subseteq P_B$ and $(x+A) \cup (y+B) \subseteq P$, for some x, $y \in \mathbb{Z}^2$, with $|P_A| \leq |A|+r$, $|P_B| \leq |B|+r$, $|P_A|+|P_B| \leq 2|B|+2r$ and $|P| \leq \frac{|A|+|B|}{2}+\frac{3}{2}r$. A similar result is proved assuming A is covered by 2 horizontal lines and B by 1 and vice versa. This generalizes a result of Stanchescu handling the case $A = B$ and extends the Freiman $3k-4$ Theorem to 2-dimensional sumsets with $|A+B| < |A|+\frac{7}{3}|B|-5$.

1. INTRODUCTION

Let G be an abelian group and let A, $B \subseteq G$ be subsets of G. We define their sumset to be

$$A + B = \{a + b : \ a \in A,\, b \in B\}.$$

Moreover, we let

$$\delta(A,B) = \begin{cases} 1 & \text{if } x + A \subseteq B \text{ for some } x \in G, \\ 0 & \text{otherwise.} \end{cases}$$

We let $\langle A\rangle$ denote the subgroup generated by A and observe that $\langle A-A\rangle$ is the minimal subgroup K such that A is contained in a K-coset. In particular, if $0 \in A$, then $\langle A-A\rangle = \langle A\rangle$. Likewise, for $G = \mathbb{Z}$, $\gcd(A-A) \in \mathbb{N}_0$ is the minimal non-negative integer d such that A is contained in an arithmetic progression with difference d, with $\gcd(A-A) = \gcd(A)$ when $0 \in A$. We let

$$\mathrm{diam}(A) = \max A - \min A$$

denote the diameter of a finite subset $A \subseteq \mathbb{Z}$. If $G = \mathbb{Z}^d$, we let $e_1, \ldots, e_d \in \mathbb{Z}^d$ be the standard basis vectors, so e_i has a 1 at the i-th coordinate and zeros elsewhere, and we let $\pi_i : \mathbb{Z}^d \to \mathbb{Z}$ be the projection onto the i-th coordinate with respect to the basis $e_1, \ldots, e_d \in \mathbb{Z}^d$.

Our starting point is the $3k-4$ Theorem. The following is the most general version of the $3k-4$ Theorem currently known. The form given below may be found in [6, Theorem 7.1 and comments thereafter] and is the result of successive contributions from Freiman [3], Lev and Smeliansky [8], Stanchescu [12], and Bardaji and Grynkiewicz [1]. Worth noting, if $|B| \leq |A|$, then the upper bound from Theorem A(i) becomes

$$|A + B| \leq |A| + 2|B| - 3 - \delta(A,B).$$

Indeed, $|A|-\delta(B,A) < |B|-\delta(A,B)$ is only possible, in view of $|A| \geq |B|$, if $|A| = |B|$, $\delta(A,B) = 0$ and $\delta(B,A) = 1$. However, it is easily noted from the definition of δ that $\delta(A,B) = \delta(B,A)$ when $|A| = |B|$, meaning this is actually never possible. Also, both hypotheses (i) and (ii) imply $|A|,\ |B| \geq 2$ (equivalent to $\operatorname{diam} A,\ \operatorname{diam} B \geq 1$) in view of basic lower bounds for $|A+B|$ (see Theorem C).

Theorem A (3k−4 Theorem)**.** *Let $A,\ B \subseteq \mathbb{Z}$ be finite and nonempty, let $d = \gcd(A+B-A-B)$ and let $|A+B| = |A|+|B|-1+r$. If either*

(i) $|A+B| \leq |A|+|B|-3+\min\{|B|-\delta(A,B),\ |A|-\delta(B,A)\}$, *or*

(ii) $\operatorname{diam} B \leq \operatorname{diam} A$, $\gcd(A-A) \leq 2d$ *and* $|A+B| \leq |A|+2|B|-3-\delta(A,B)$,

then there are arithmetic progressions P, Q and R of common difference d with

$$A \subseteq P, \quad B \subseteq Q, \quad |P \setminus A| \leq r, \quad |Q \setminus B| \leq r, \tag{1}$$
$$R \subseteq A+B \ \text{ and } \ |R| \geq |A|+|B|-1.$$

Moreover, if either (i) or (ii) holds with $\operatorname{diam} B \leq \operatorname{diam} A$ and $|B| \geq |A|$, then

$$|Q \setminus B| \leq r - (|B|-|A|).$$

In short, the $3k-4$ theorem shows that a sumset $A+B \subseteq \mathbb{Z}$ with small sumset below the threshold $|A+B| \leq |A|+|B|-3+\min\{|B|-\delta(A,B),\ |A|-\delta(B,A)\}$ is only possible if A, B and $\mathbb{Z} \setminus (A+B)$ can all be approximated by arithmetic progressions P, Q and R with common difference, in the sense that the "distance" between each set and the respective set P, Q and $\mathbb{Z} \setminus R$, as measured by inclusion, is small. Note that the set $A+B$ containing a long arithmetic progression R is equivalent to its complement being contained in the complement of R with $|\big(\mathbb{Z} \setminus (A+B)\big) \setminus \big(\mathbb{Z} \setminus R\big)| \leq r$. Thus the theorem is symmetric with regards to all three sets A, B and $A+B$. Moreover, the bound r for the number of holes separating the individual sets from their progressions is known to be precise, as is the threshold hypothesis $|A+B| \leq |A|+|B|-3+\min\{|B|-\delta(A,B),\ |A|-\delta(B,A)\}$. In both cases, there are examples showing these bounds cannot be improved. For instance, if $r \geq 0$, $a \geq r+1$ and $b \geq r+1$ with strict inequality in at least one of the latter two inequalities, then the sets

$$A = [0, a-1-r] \cup \{a-r+1, a-r+3, \ldots, a-r+(2r-1)\} \quad \text{and}$$
$$B = [0, b-1-r] \cup \{b-r+1, b-r+3, \ldots, b-r+(2r-1)\}$$

have $|A| = a$, $|B| = b$, and

$$A+B = [0, a+b-2] \cup \{a+b, a+b+2, \ldots, a+b+2r-2\}$$

with $|A+B| = |A|+|B|-1+r \leq |A|+|B|-2+\min\{|B|-\delta(A,B),\ |A|-\delta(B,A)\}$, and it is easily seen that $|P \setminus A| = |Q \setminus B| = |\big(\mathbb{Z} \setminus R\big) \setminus \big(\mathbb{Z} \setminus (A+B)\big)| = r$, where $P = [0, a-1+r]$, $Q = [0, b-1+r]$ and $R = [0, a+b-2]$, cannot be improved upon. Likewise, if $A = I_1 \cup I_2$ is the union of two intervals I_1 and I_2 with $\min I_2 - \max I_1$ sufficiently large and $|A| \geq 3$, then

$|A+A| = (2|I_1|-1)+(|I_1|+|I_2|-1)+(2|I_2|-1) = 3|A|-3$ with $|P\setminus A|$ unbounded. Alternatively, taking $B = J$ to be a single interval J with $\min I_2 - \max I_1$ sufficiently large and $|A| \geq 3$ gives $|A+B| = (|I_1|+|J|-1)+(|I_2|+|J|-1) = |A|+2|B|-2$ with $|P \setminus A|$ again unbounded.

As the latter examples above show, there is no way to approximate the summands in a small sumset $A+B \subseteq \mathbb{Z}$ by arithmetic progressions once $|A+B|$ becomes too large. However, the problematic examples above are quite limited, being instead the union of two arithmetic progressions. This is a special case of more general framework related to Fremain's Theorem. As the first example shows, the $3k-4$ Theorem is actually one of the few instances in which the constants in Freiman's Theorem are known precisely, and for distinct summands as well. For a fuller discussion of the framework related to Freiman's Theorem, see [15] [9] [11]. The goal of this paper is to extend the precise estimates of the $3k-4$ Theorem to certain 2-dimensional sumsets under a more generous threshold bound for $|A+B|$.

If G is an abelian group and A, $B \subseteq G$ are nonempty subsets, then we can usually translate the sets A and B in any way and not significantly alter the structure of A, B or $A+B$. To simplify notation, we then translate A and B so that $0 \in A \cap B$. In this framework, sumsets, like other categorical objects, have an associated notion of morphism. A map $\psi : A+B \to G'$, where G' is another abelian group, is called a normalized Freiman homomorphism if $\psi(x+y) = \psi(x)+\psi(y)$ for all $x \in A$ and $y \in B$ (which implies $\psi(0) = 0$). Note $0 \in A \cap B$ ensures that A, $B \subseteq A+B$ are in the domain of ψ, so this definition makes sense (in general, the key requirement is that there is a common element $z \in A \cap B$ that can be used as a base point to the homomorphism, but it simplifies notation to further translate so that the common element is equal to $z = 0$). The image $\psi(A+B) = \psi(A)+\psi(B)$ is the homomorphic image of $A+B$, and the Freiman homomorphism ψ induces an isomorphism with its image, $A+B \cong \psi(A)+\psi(B)$, when ψ is injective *on all of* $A+B$ (not just on A and B, which is in general too weak of a requirement).

With the notion of Freimain isomorphism in hand, it is possible to speak of the intrinsic dimension of a sumset $A+B \subseteq G$, where $0 \in A \cap B$, independent of the group G in which $A+B$ is embedded. We define $\dim^+(A+B)$ to be the maximal integer $d \geq 0$ such that there is an injective normalized Freiman homomorphism $\psi : A+B \to G'$ with $\langle \psi(A)+\psi(B) \rangle = G'$ and G' having torsion-free rank $\mathrm{rk}(G') = d$. This dimension is independent of the translates of A and B chosen and, moreover, when the initial group G is torsion-free, there is an injective normalized Freiman homomorphism $\psi : A+B \to \mathbb{Z}^d$ with $\langle \psi(A)+\psi(B) \rangle = \mathbb{Z}^d$. See [6, Chapter 20] for details (derived via universal ambient groups). By a modification of the argument of [6, Proposition 3.1], one also deduces that there exist injective normalized Freiman homomorphisms $\psi : A+B \to \mathbb{Z}^{d'}$ with $\langle \psi(A)+\psi(B) \rangle = \mathbb{Z}^{d'}$ for any $d' \in [1,d]$ as well.

When $A+B$ is a torsion-free sumset (where A and B are finite and nonempty), meaning it has an embedding into a torsion free group, a result of Ruzsa [10] (combined with the remarks of the previous paragraph) shows that large dimension implies large sumset in the following sense:

$\dim^+(A+B) \geq d$ with $|A| \geq |B|$ implies

$$|A+B| \geq |A| + d|B| - \frac{1}{2}d(d+1).$$

For refinements to this result, see [15, Section 5.2] [7]. In particular, we see that that any finite, nonempty sumset $A+B \subseteq \mathbb{Z}$ with $\dim^+(A+B) \geq 2$ and $|A| \geq |B|$ has $|A+B| \geq |A|+2|B|-3$, which corresponds roughly to the point after which the $3k-4$ Theorem breaks down. However, $\dim^+(A+B) \geq 3$ and $|A| \geq |B|$ gives a bound of $|A+B| \geq |A|+3|B|-6$, meaning the only obstacle to extending the $3k-4$ Theorem upwards towards the threshold $|A|+3|B|-7$ are 1- and 2-dimensional sumsets. For 2-dimensional sumsets, there is a refinement [7] of the result of Ruzsa.

Theorem B. *Let $s \geq 2$ be an integer. Let $A, B \subseteq \mathbb{R}^2$ be finite subsets with $|A| \geq |B| \geq 2s^2-3s+2$. If*

$$|A+B| < |A| + (3-\frac{2}{s})|B| - 2s + 1,$$

then there is a line ℓ such that each of A and B can be covered by at most $s-1$ parallel translates of ℓ.

In particular (taking $s=3$), when $|A| \geq |B| \geq 11$ and $|A+B| < |A| + \frac{7}{3}|B| - 5$, both A and B are covered by at most 2 parallel lines, meaning the only obstacles to extending the $3k-4$ Theorem upwards towards the threshold $|A|+\frac{7}{3}|B|-5$ are 1-dimensional sumsets as well as 2-dimensional sumsets with both summands covered by at most 2 parallel lines. The latter sets are the subject of this paper. Indeed, our main results are the following, which extend the $3k-4$ Theorem to 2-dimensional sumsets with both summands covered by at most 2 parallel lines. Note we have normalized the hypotheses below so that these lines run parallel to e_1 with $A+B$ generating $\mathbb{Z}^2$ affinely. Also, $|\pi_2(A)|$ simply counts the number of lines parallel to $\mathbb{Z}e_1$ that are needed to cover A.

Theorem 1.1. *Let $A, B \subseteq \mathbb{Z}^2$ be finite, nonempty subsets with $|\pi_2(B)| = 1$ and $|\pi_2(A)| = 2$. Suppose $\langle A+B-A-B \rangle = \mathbb{Z}^2$ and*

$$|A+B| = |A| + 2|B| - 2 + r \leq |A| + 2|B| + \min\{|A|-1, |B|\} - 5.$$

Then there exist subsets $P_A, P_B \subseteq \mathbb{Z}^2$, with P_B an arithmetic progression with difference e_1, and P_A the union of two arithmetic progressions with difference e_1, such that $B \subseteq P_B$, $A \subseteq P_A$,

$$|P_A \setminus A| \leq r \quad \text{and} \quad |P_B \setminus B| \leq r.$$

Theorem 1.2. *Let $A, B \subseteq \mathbb{Z}^2$ be finite, nonempty subsets with $|\pi_2(A)| = |\pi_2(B)| = 2$. Suppose $\langle A+B-A-B \rangle = \mathbb{Z}^2$, $|A| \geq |B|$ and*

$$|A+B| = |A| + 2|B| - 2 + r - \delta(A,B) \leq 2|A| + 2|B| - 6 - \delta(A,B).$$

Then there exist subsets P_A, $P_B \subseteq \mathbb{Z}^2$, each the union of two arithmetic progressions with difference e_1, such that $A \subseteq P_A$, $B \subseteq P_B$,

$$|P_A \setminus A| \leq r \quad \text{and} \quad |P_B \setminus B| \leq r.$$

Theorem 1.3. *Let A, $B \subseteq \mathbb{Z}^2$ be finite, nonempty subsets with $|\pi_2(A)| = |\pi_2(B)| = 2$. Suppose $\langle A+B-A-B\rangle = \mathbb{Z}^2$, $|A| \geq |B|$, $|A+B| \leq |A| + \frac{19}{7}|B| - 5$ and*

$$|A+B| = |A| + 2|B| - 2 - \delta(A,B) + r = |A| + |B| + \frac{|A|+|B|}{2} - 3 + r'.$$

Then there exist subsets P_A, P_B, $P \subseteq \mathbb{Z}^2$, each the union of two arithmetic progressions with difference e_1, such that $A \subseteq P_A$, $B \subseteq P_B$, $(x+A) \cup (y+B) \subseteq P$ for some $x, y \in \mathbb{Z}^2$,

$$|P_A \setminus A| \leq r, \quad |P_B \setminus B| \leq r, \quad |P_A \setminus A| + |P_B \setminus B| \leq 2r', \quad \text{and}$$
$$|P \setminus (x+A)| + |P \setminus (y+B)| \leq 2r + 2 + \Big||P_A| - |P_B|\Big| - \Big||A| - |B|\Big| \leq 3r + 2.$$

Moreover, $|P \setminus (x+A)| + |P \setminus (y+B)| \leq 2r'$ unless

$$y + P_B \subseteq x + P_A = P \quad \text{and} \quad |P \setminus (x+A)| + |P \setminus (y+B)| = 2|P_A \setminus A| + |A| - |B|.$$

As with the $3k-4$ Theorem, the bounds in the above theorems are precise (we will give examples later in the paper). Theorem 1.3 generalizes the two-dimensional case of [13] [14], which handles the symmetric case when $A = B$. We use many of the same compression methods of [12] [14] to reduce the 2-dimensional sumset in consideration to several 1-dimensional sumsets that can then be dealt with via the $3k-4$ Theorem. However, unlike the case when $A = B$, much of the added difficulty in the proof of Theorem 1.3 will arise from trying to show A and B can be approximated simultaneously by the same progression P, a fact which holds trivially when $A = B$, and which clearly cannot hold in Theorem 1.1.

2. Refined Theory Involving the $3k-4$ Theorem

There are some important consequences of Theorem A that "refine" its use in practice. These details are all contained in [1], but we repeat them here owing to their critical role in the proof of Theorem 1.3. Given a set $X \subseteq \mathbb{Z}$, we let $P_X \subseteq \mathbb{Z}$ denote the minimal arithmetic progression with difference 1 containing X. Let A, $B \subseteq \mathbb{Z}$ be finite, nonempty subsets with $\langle A+B-A-B\rangle = \mathbb{Z}^2$,

$$\text{diam}(B) \leq \text{diam}(A) \leq |A| + |B| - 3 \quad \text{and} \quad |A+B| \leq |A| + 2|B| - 3 - \delta(A,B). \tag{2}$$

Moreover, let $|A+B| = |A|+|B|-1+r$. These assumptions will apply throughout the discussion of this section.

The quantity $|P_A \setminus A|$ counts the number of "holes" in A with respect to P_A, and it is easily noted that $\text{diam}\, A = |A| + |P_A \setminus A| - 1$. There are various ways to obtain the hypothesis

$\operatorname{diam} A \leq |A|+|B|-3$, which is equivalent to $|P_A \setminus A| \leq |B|-2$ in view of the previous equality. For instance, if

$$\gcd(A-A) \leq 2, \quad \operatorname{diam}(B) \leq \operatorname{diam}(A) \quad \text{and} \quad |A+B| \leq |A|+2|B|-3-\delta(A,B), \tag{3}$$

then Theorem A(ii) implies that

$$|A+B| \geq |A|+|B|-1+\max\{|P_A \setminus A|,\, |P_B \setminus B|\}, \tag{4}$$

the above being a useful and equivalent reformulation of (1). Combined with the upper bound for $|A+B|$ from (3), it then follows that $|P_A \setminus A| \leq |B|-2-\delta(A,B)$. In summary, the hypotheses from (3) imply those of (2) via Theorem A and will be the usual way that we obtain the setup found in (2).

Regardless, assuming A and B satisfy (2), we have the existence of an arithmetic progression $R \subseteq \mathbb{Z}$ with difference 1 such that

$$R \subseteq A+B \quad \text{and} \quad |R| \geq |A|+|B|-1$$

per the main result of [1]. We also have $R \subseteq P_{A+B}$ with $|P_{A+B}| = |A|+|B|-1+|P_A \setminus A|+|P_B \setminus B|$, meaning R covers all but $|P_A \setminus A|+|P_B \setminus B|$ elements of the interval P_{A+B}. Assuming (4) holds (as would be the case under the assumptions of (3)), we find that

$$|P_{A+B} \setminus (A+B)| \leq \min\{|P_A \setminus A|,\, |P_B \setminus B|\}.$$

The existence of the interval $R \subseteq A+B$ with $|R| \geq |A|+|B|-1$ has several important consequences, all derived in [1]. First,

$$\min P_{A+B} + \Big[|P_A \setminus A|+|P_B \setminus B|, |A|+|B|-2\Big] \subseteq A+B,$$

for there are simply not enough elements in P_{A+B} for the arithmetic progression R to avoid this interval no matter where $R \subseteq P_{A+B}$ occurs. Since $[\min A+\max B-1, \min B+\max A+1]$ is contained in the above interval (given the assumptions of (2)), a particular consequence is that

$$[\min A+\max B-1, \min B+\max A+1] \subseteq A+B.$$

Thus all elements of $P_{A+B} \setminus (A+B)$ are contained in $P_{\min A+B} \cup P_{\max A+B}$. In particular, if $x \in P_{A+B} \setminus (A+B)$, then either

$$-\min A + x \in P_B \setminus B \quad \text{or} \quad -\max A + x \in P_B \setminus B.$$

An element $x \in P_{A+B} \setminus (A+B)$ of the first type is called a left hole in $A+B$, and those of the second type are called right holes. Equivalently, since $[\min A+\max B-1, \min B+\max A+1] \subseteq R \subseteq A+B$, it follows that a hole $x \in P_{A+B} \setminus (A+B)$ is left if $x < \min R$ and is right if $x > \max R$, which means that if x is a left hole and y is a right hole, then $y-x \geq (\max R+1)-(\min R-1) \geq |A|+|B|$.

Likewise, an element $x \in P_B \setminus B$ with $\min A+x \in P_{A+B} \setminus (A+B)$ is called a left stable hole in B, an element $x \in P_B \setminus B$ with $\max A+x \in P_{A+B} \setminus (A+B)$ is called a right stable hole in B, and all other $x \in P_B \setminus B$ are called unstable holes. Observing that $(\max A + x) - (\min A + x) = \operatorname{diam} A \le |A| + |B| - 3$, we conclude that no stable hole in B can be both left and right stable. Moreover, if $x \in P_B \setminus B$ is left stable and $y \in P_B \setminus B$ is right stable, then $(y + \max A) - (x + \min A) \ge |A| + |B|$, which implies $y - x \ge |B| - |P_A \setminus A| + 1 \ge 3$. Thus all left stable holes precede all right stable holes in B. Similar definitions and conclusions hold regarding A: an element $x \in P_A \setminus A$ with $\min B + x \in P_{A+B} \setminus (A + B)$ is called a left stable hole in A, an element $x \in P_A \setminus A$ with $\max B + x \in P_{A+B} \setminus (A + B)$ is right stable, and all left stable holes precede right stable ones in A.

Let e be the greatest left stable hole in B (or $\min B - 1$ if none exist) and let c be the smallest right stable hole in B (or $\max B + 1$ if none exist). Then $J_B = [e+1, c-1] \subseteq P_B$ is a nonempty interval. Moreover, $A + J_B \subseteq P_A + J_B = [\min A + e + 1, \max A + c - 1] \subseteq A + B$ per definition of left and right holes and the extremal assumptions on e and c, which means $A + (B \cup J_B) = A + B$. Noticing that the left stable hole $x \in P_B \setminus B$ corresponds to the left hole $\min A + x \in P_{A+B} \setminus (A + B)$ and then to the left stable hole $\min A - \min B + x \in P_A \setminus A$, with similar statements holding for right holes, we find that $J_A = [\min A - \min B + e + 1, \max A - \max B + c - 1] \subseteq P_A$ has $(A \cup J_A) + (B \cup J_B) = A + B$. This has important consequences. For instance, if we were applying the $3k - 4$ Theorem to $A + B$, then we can instead imply it to $(A \cup J_A) + (B \cup J_B)$ resulting in better bounds. Indeed, (4) then improves by one for each element of $J_A \setminus A$ and $J_B \setminus B$:

$$|A + B| \ge |A| + |B| - 1 + \max\{|P_A \setminus A| + |J_B \setminus B|,\ |P_B \setminus B| + |J_A \setminus A|\}. \tag{5}$$

Likewise the bound on the size of $|R|$ increases to

$$|R| \ge |A| + |B| - 1 + |J_B \setminus B| + |J_A \setminus A|,$$

with corresponding improvements in other associated bounds mentioned above.

By definition of e and c, we have $\min A + e \notin A + B$ and $\max A + c \notin A + B$. Thus the progression R must lie wholly in one of the intervals $[\min A + \min B, \min A + e - 1]$, $[\max A + c + 1, \max A + \max B]$ or $[\min A + e + 1, \max A + c - 1]$. Since the first two interval have respective sizes $e - \min B < \max B - \min B = \operatorname{diam} B \le \operatorname{diam} A \le |A| + |B| - 3 < |R|$ and $\max B - c < \max B - \min B \le |A| + |B| - 3 < |R|$, we conclude that $R \subseteq [\min A + e + 1, \max A + c - 1]$, implying that $\operatorname{diam} A - 1 + c - e = |A| + |P_A \setminus A| - 1 + |J_B| \ge |R| \ge |A| + |B| - 1 + |J_B \setminus B| + |J_A \setminus A|$. Thus

$$|J_B| \ge |B| - |P_A \setminus A| + |J_A \setminus A| + |J_B \setminus B|,$$

and

$$|J_A| = |J_B| + \operatorname{diam} A - \operatorname{diam} B \ge |A| - |P_B \setminus B| + |J_A \setminus A| + |J_B \setminus B|.$$

3. Setup and Proof of Theorems 1.1, 1.2 and 1.3

We will frequently use the following basic and easily proven bound for torsion free sumsets (see [6, Theorem 3.1]).

Theorem C. *Let G be a torsion free abelian group and let A, $B \subseteq G$ be finite and nonempty subsets. Then*

$$|A+B| \geq |A|+|B|-1.$$

A simple corollary of the above result is the following.

Corollary 3.1. *Let A, $B \subseteq \mathbb{Z}$ be finite, nonempty subsets and let $d \geq 1$. Suppose $d \mid \gcd(B-B)$ and let $s \in [1,d]$ denote the number of $d\mathbb{Z}$-cosets that intersect A. Then*

$$|A+B| \geq |A|+s(|B|-1).$$

In particular, if $\gcd(B-B) \neq 1$ and $\gcd(A-A)=1$, then $|A+B| \geq |A|+2|B|-2$.

Proof. Let $x_1, \ldots, x_s \in A$ be a set of representatives for the $d\mathbb{Z}$-cosets that intersect A. These cosets give rise to a partition $A = \bigcup_{i=1}^{s} A_i$, where $A_i = (x_i + d\mathbb{Z}) \cap A \neq \emptyset$. The hypothesis $d \mid \gcd(B-B)$ ensures that B is itself contained in a single $d\mathbb{Z}$-coset. Thus the $A_i + B$ are each contained in distinct $d\mathbb{Z}$-cosets, implying $|A+B| = \sum_{i=1}^{s} |A_i + B|$. Applying Theorem C to each $|A_i + B|$ yields the desired bound. Finally, if $d = \gcd(B-B) > 1$ and $\gcd(A-A) = 1$, then $s \geq 2$ follows, and now the previous bound implies $|A+B| \geq |A|+2|B|-2$. This bound is trivial when $|B| = 1$, i.e., when $\gcd(B-B) = 0$. □

We next introduce several notions for 2-dimensional sets. The first is that of compression, a concept that has been exploited to much success in additive theory [2] [4] [5] [6, Chapter 7] [7] [14] .

Let $A \subseteq \mathbb{Z}^2$ be a finite subset. The linear compression of A, with respect to e_2 and denoted $\mathbf{C}_{e_2}(A)$, is the set obtained by compressing and shifting A along each vertical line until the resulting set is an arithmetic progression with difference e_2 whose first term is contained on the horizontal axis. More concretely, we define the set $\mathbf{C}_{e_2}(A) \subseteq \mathbb{Z}^2$ piecewise by it's intersections with the lines $\mathbb{Z}e_2 + xe_1$, for $x \in \mathbb{Z}$, by letting $\mathbf{C}_{e_2}(A) \cap (\mathbb{Z}e_2 + xe_1)$ be the subset of $\mathbb{Z}e_2 + xe_1$ satisfying

$$\pi_2(\mathbf{C}_{e_2}(A) \cap (\mathbb{Z}e_2 + xe_1)) = \{0, 1, 2, \ldots, (r-1)\},$$

where $|A \cap (\mathbb{Z}e_2 + xe_1)| = r$ and the right hand side is considered empty if $r = 0$. Letting $C_t := C \cap (\mathbb{Z}e_2 + te_1)$ below, it follows in view of Theorem C that

$$\begin{aligned} |A+B| &= \sum_{t\in\mathbb{Z}} |(A+B)_t| \\ &\geq \sum_{t\in\mathbb{Z}} \max\{|A_s + B_{t-s}| :\ A_s \neq \emptyset, B_{t-s} \neq \emptyset\} \\ &\geq \sum_{t\in\mathbb{Z}} \max\{|A_s| + |B_{t-s}| - 1 :\ A_s \neq \emptyset, B_{t-s} \neq \emptyset\} \\ &= |\mathbf{C}_{e_2}(A) + \mathbf{C}_{e_2}(B)|, \end{aligned} \tag{6}$$

for finite subsets A, $B \subseteq \mathbb{Z}^2$.

For Theorems 1.1, 1.2 and 1.3, we have $|\pi_2(A)| = 2$ and $|\pi_2(B)| \leq 2$. By translating appropriately, we may assume

$$A = A_0 \cup A_1 \quad \text{and} \quad B = B_0 \cup B_1$$

with A_0, $B_0 \subseteq \mathbb{Z}e_1$ both nonempty and A_1 and B_1 the elements of A and B, respectively, not contained on the horizontal line $\mathbb{Z}e_1$. Observe this means all elements of A_1 lie on a horizontal line parallel to $\mathbb{Z}e_1$, say on $\mathbb{Z}e_1 + xe_2$, as do all elements of B_1, say on $\mathbb{Z}e_1 + ye_2$. Thus B_1 is nonempty precisely when $|\pi_2(B)| = 2$ rather than $|\pi_2(B)| = 1$. If $|\pi_2(B)| = 2$ and $|x| \neq |y|$, then applying Theorem C four times gives $|A+B| \geq |A_0+B_0|+|A_0+B_1|+|A_1+B_0|+|A_1+B_1| \geq (|A_0|+|B_0|-1)+(|A_0|+|B_1|-1)+(|A_1|+|B_0|-1)+(|A_1|+|B_1|-1) = 2|A|+2|B|-4$, contrary to hypothesis. Therefore we may assume $|x| = |y|$, which together with $\langle A+B-A-B\rangle = \mathbb{Z}^2$ forces $|x| = |y| = 1$, and now, replacing A by $A+e_2$ if $x = -1$ and B by $B+e_2$ if $y = -1$, we may assume

$$A_0,\, B_0 \subseteq \mathbb{Z}e_1, \quad A_1,\, B_1 \subseteq \mathbb{Z}e_1 + e_2 \quad \text{and} \quad A_0,\, B_0,\, A_1 \neq \emptyset.$$

For any set $X \subseteq \mathbb{Z}^2$ with $|\pi_2(X)| = j \leq 2$, there is a unique minimal (by inclusion) set $P_X \subseteq \mathbb{Z}^2$ that is the union of j arithmetic progressions with difference e_1 and has $X \subseteq P_X$. Thus P_X minimizes $|P_X \setminus X|$ over all P_X that are the union of j arithmetic progressions with difference e_1. We will use this notation P_X for the remainder of the paper.

Leaving A_0 and B_0 fixed but translating A_1 and B_1 by the same constant αe_1, where $\alpha \in \mathbb{Z}$, has the effect of shifting both the sets A_1 and B_1 along the horizontal line $\mathbb{Z}e_1 + e_2$ by the amount α. This is the same as applying the linear transformation $\psi : \mathbb{Z}^2 \to \mathbb{Z}^2$ given by $(x, y) \mapsto (x + \alpha y, y)$, which is easily seen to have determinant 1 and thus map $\mathbb{Z}^2$ isomorphically onto $\mathbb{Z}^2$. We will call such a linear transformation ψ a *horizontal shift*. Using a horizontal shift by α, we can replace the sets A_1 and B_1 by $A_1 + \alpha e_1$ and $B_1 + \alpha e_1$, in effect, allowing us to slide the sets A_1 and B_1 by the same amount along the line $\mathbb{Z}e_1 + e_2$. Since it is readily seen that the desired conclusions holding for the sumset $\psi(A) + \psi(B)$ imply that the desired conclusions hold for the original sumset $A + B$, we will make free use of horizontal shifts in the proofs below.

For instance, given any fixed $x \in \{0, 1\}$, it is easily seen that there is a horizontal shift $\psi : \mathbb{Z}^2 \to \mathbb{Z}^2$ so that

$$\max \pi_1(\psi(A_x)) \leq \min \pi_1(\psi(A_{1-x})) \quad \text{and} \quad \max \pi_1(\psi(B_x)) \leq \min \pi_1(\psi(B_{1-x})), \tag{7}$$

where $\max \emptyset := +\infty$ and $\min \emptyset := -\infty$, with equality holding in at least one of the estimates in (7). Indeed, to see this, simply choose a horizontal shift such that equality holds in the first inequality of (7). Then, if the second inequality in (7) fails, further shift the sets A_{1-x} and B_{1-x} to the right until equality holds in the second inequality of (7). As further shifting A_{1-x} only increases $\min \pi_1(\psi(A_{1-x}))$, the first inequality of (7) remains true, showing (7) holds.

Let $\widetilde{A} := \mathbf{C}_{e_2}(\psi(A))$ and $\widetilde{B} := \mathbf{C}_{e_2}(\psi(B))$, with $\widetilde{A} = \widetilde{A}_0 \cup \widetilde{A}_1$ and $\widetilde{B} = \widetilde{B}_0 \cup \widetilde{B}_1$, where $\widetilde{A}_i = \widetilde{A} \cap (\mathbb{Z}e_1 + ie_2)$ and $\widetilde{B}_i = \widetilde{B} \cap (\mathbb{Z}e_1 + ie_2)$ for $i \in \{0,1\}$. Since $\langle A + B - A - B\rangle = \mathbb{Z}^2$ with $\psi : \mathbb{Z}^2 \to \mathbb{Z}^2$ an isomorphism, it follows that

$$\langle \psi(A + B - A - B)\rangle = \mathbb{Z}^2.$$

Moreover, we have

$$\langle \widetilde{A}_0 + \widetilde{B}_0 - \widetilde{A}_0 - \widetilde{B}_0\rangle = \mathbb{Z}e_1, \tag{8}$$

for if instead $\langle \widetilde{A}_0 + \widetilde{B}_0 - \widetilde{A}_0 - \widetilde{B}_0\rangle = d\mathbb{Z}e_1$ with $d \geq 2$, then it is easily seen that $\langle \psi(A) - \psi(A)\rangle \leq \langle \widetilde{A}_0 - \widetilde{A}_0\rangle \times \mathbb{Z} \leq \langle \widetilde{A}_0 + \widetilde{B}_0 - \widetilde{A}_0 - \widetilde{B}_0\rangle \times \mathbb{Z} = \mathbb{Z}de_1 \times \mathbb{Z}$ (the first inclusion follows in view of $|\widetilde{A}_1| \leq 1$; see (10) below) and $\langle \psi(B) - \psi(B)\rangle \leq \langle \widetilde{B}_0 - \widetilde{B}_0\rangle \times \mathbb{Z} \leq \langle \widetilde{A}_0 + \widetilde{B}_0 - \widetilde{A}_0 - \widetilde{B}_0\rangle \times \mathbb{Z} = \mathbb{Z}de_1 \times \mathbb{Z}$ (the first inclusion follows in view of $|\widetilde{B}_1| \leq 1$), contradicting that $\langle \psi(A) - \psi(A)\rangle + \langle \psi(B) - \psi(B)\rangle = \langle \psi(A) + \psi(B) - \psi(A) - \psi(B)\rangle = \mathbb{Z}^2$. The following properties are also easily observed:

$$|\widetilde{A}| = |A| \qquad \text{and} \qquad |\widetilde{B}| = |B|, \tag{9}$$

$$|\widetilde{A}_1|,\, |\widetilde{B}_1| \leq 1 \qquad \text{and} \qquad \max\{|\widetilde{A}_1|,\, |\widetilde{B}_1|\} = 1, \tag{10}$$

$$|P_{\widetilde{A}} \setminus \widetilde{A}| = |P_{\widetilde{A}_0} \setminus \widetilde{A}_0| \qquad \text{and} \qquad |P_{\widetilde{B}} \setminus \widetilde{B}| = |P_{\widetilde{B}_0} \setminus \widetilde{B}_0| \tag{11}$$

$$|P_A \setminus A| \leq |P_{\widetilde{A}} \setminus \widetilde{A}| \qquad \text{and} \qquad |P_B \setminus B| \leq |P_{\widetilde{B}} \setminus \widetilde{B}|, \tag{12}$$

$$|A + B| \quad \geq \quad |\widetilde{A} + \widetilde{B}|, \tag{13}$$

where (13) follows in view of (6). The use of such pairs $(\widetilde{A}, \widetilde{B})$ to study more general configurations of points in the plane follows that of Freiman [4] and Stanchescu [14]. They will allow us to attain the desired bounds for $|P_A \setminus A|$ and $|P_B \setminus B|$ rather easily.

Proof of Theorem 1.1. Let $\widetilde{A} = \widetilde{A}_0 \cup \widetilde{A}_1$ and $\widetilde{B} = \widetilde{B}_0 \cup \widetilde{B}_1$ be as defined above. Then, since $B_1 = \emptyset$, it follows that $\widetilde{B}_1 = \emptyset$, $\widetilde{B}_0 = \widetilde{B}$, $|\widetilde{A}_1| = 1$ and $|\widetilde{A}_0| = |\widetilde{A}| - 1$. Consequently, it follows from (9) and (13) that

$$|\widetilde{A}_0 + \widetilde{B}| + |\widetilde{B}| = |\widetilde{A} + \widetilde{B}| \leq |A + B| = |\widetilde{A}_0| + 2|\widetilde{B}| - 1 + r \leq |\widetilde{A}_0| + 2|\widetilde{B}| + \min\{|\widetilde{A}_0|,\, |\widetilde{B}|\} - 4.$$

Thus, in view of (8), we can apply Theorem A(i) (with $d = 1$) to $\widetilde{A}_0 + \widetilde{B}$ to conclude $|P_{\widetilde{A}_0} \setminus \widetilde{A}_0| \leq r$ and $|P_{\widetilde{B}} \setminus \widetilde{B}| \leq r$, and now the proof is complete in view of (12) and (11). □

Proof of Theorem 1.2. Let $\widetilde{A} = \widetilde{A}_0 \cup \widetilde{A}_1$ and $\widetilde{B} = \widetilde{B}_0 \cup \widetilde{B}_1$ be as defined above. From our hypotheses and (13) and (9), we find that

$$|\widetilde{A} + \widetilde{B}| \leq |A + B| \leq |\widetilde{A}| + 2|\widetilde{B}| - 2 + r \leq |\widetilde{B}| + 2|\widetilde{A}| - 2 + r$$

and

$$|\widetilde{A} + \widetilde{B}| \leq |A + B| \leq 2|\widetilde{A}| + 2|\widetilde{B}| - 6.$$

Thus, if either $\widetilde{B}_1 = \emptyset$ or $\widetilde{A}_1 = \emptyset$, we can apply Theorem 1.1 to $\widetilde{A} + \widetilde{B}$ to conclude $|P_{\widetilde{A}} \setminus \widetilde{A}| \leq r$ and $|P_{\widetilde{B}} \setminus \widetilde{B}| \leq r$, in which case the proof is complete in view of (12). Therefore, in view of (10), and by appropriately translating the sets A and B, we may instead assume $\widetilde{A}_1 = \widetilde{B}_1 = \{0\}\times\{1\}$, in which case

$$|\widetilde{A}_0| = |A| - 1 \geq |B| - 1 = |\widetilde{B}_0|.$$

Note

$$\begin{aligned} |A+B| \geq |\widetilde{A}+\widetilde{B}| &= |\widetilde{A}_0+\widetilde{B}_0| + |(0+\widetilde{B}_0)\cup(\widetilde{A}_0+0)| + 1 \\ &= |\widetilde{A}_0+\widetilde{B}_0| + |\widetilde{B}_0| + |\widetilde{A}_0| - |\widetilde{B}_0\cap\widetilde{A}_0| + 1. \end{aligned} \tag{14}$$

We have the trivial estimate

$$|\widetilde{B}_0\cap\widetilde{A}_0| \leq |\widetilde{B}_0|, \tag{15}$$

with equality only possible if $\widetilde{B}_0 \subseteq \widetilde{A}_0$. We also have the trivial estimate

$$|\widetilde{B}_0\cap\widetilde{A}_0| \leq |\widetilde{A}_0| \tag{16}$$

with equality only possible if $\widetilde{A}_0 = \widetilde{B}_0$ (in view of $|\widetilde{A}_0| \geq |\widetilde{B}_0|$). However, in view of our normalization assumption $\widetilde{A}_1 = \widetilde{B}_1 = \{0\}\times\{1\}$ and the definition of $\widetilde{A}$ and $\widetilde{B}$, it follows that the equality $\widetilde{A}_0 = \widetilde{B}_0$ is only possible if $A = B$. In particular, it is only possible if $\delta(A,B) = 1$, allowing us to refine the estimate in (16) to

$$|\widetilde{B}_0\cap\widetilde{A}_0| \leq |\widetilde{A}_0| - 1 + \delta(A,B). \tag{17}$$

By hypothesis, we have

$$|A+B| = |\widetilde{A}| + 2|\widetilde{B}| - 2 + r - \delta(A,B) = |\widetilde{A}_0| + 2|\widetilde{B}_0| + 1 + r - \delta(A,B) \quad \text{and} \tag{18}$$

$$|A+B| \leq 2|\widetilde{A}| + 2|\widetilde{B}| - 6 - \delta(A,B) = 2|\widetilde{A}_0| + 2|\widetilde{B}_0| - 2 - \delta(A,B). \tag{19}$$

Combining (14), (17) and (18) yields

$$|\widetilde{A}_0+\widetilde{B}_0| \leq |\widetilde{A}_0| + |\widetilde{B}_0| - 1 + r.$$

Combining (14), (15) and (19) yields

$$|\widetilde{A}_0+\widetilde{B}_0| \leq |\widetilde{A}_0| + 2|\widetilde{B}_0| - 3 - \delta(A,B),$$

with strict inequality unless $\widetilde{B}_0 \subseteq \widetilde{A}_0$. If we have have strict inequality, then

$$|\widetilde{A}_0+\widetilde{B}_0| \leq |\widetilde{A}_0| + 2|\widetilde{B}_0| - 3 - \delta(\widetilde{A}_0,\widetilde{B}_0). \tag{20}$$

On the other hand, if $\widetilde{B}_0 \subseteq \widetilde{A}_0$, then $\delta(\widetilde{A}_0,\widetilde{B}_0) = 1$ (which implies $|\widetilde{A}_0| \leq |\widetilde{B}_0|$) is only possible if $\widetilde{A}_0 = \widetilde{B}_0$, further implying $A = B$ and $\delta(A,B) = 1$. Thus $\delta(A,B) \geq \delta(\widetilde{A}_0,\widetilde{B}_0)$, so that (20) holds in this case as well. Consequently, in view of (8), we can apply Theorem A(i) (with $d = 1$) to $\widetilde{A}_0+\widetilde{B}_0$ to conclude $|P_{\widetilde{A}_0}\setminus\widetilde{A}_0| \leq r$ and $|P_{\widetilde{B}_0}\setminus\widetilde{B}_0| \leq r$, and now the proof is complete in view of (12) and (11). □

We now come to the proof of our main theorem.

Proof of Theorem 1.3. The assumption $|A| \geq |B|$ is present in the statement of Theorem 1.3 solely to simplify its presentation. However, to take advantage of symmetry in what follows, we will *not* assume this in the proof. Instead, let $\widetilde{A}$ be a set from among A and B with larger cardinality and let $\widetilde{B}$ be the other set. By hypothesis,

$$|A+B| \leq |\widetilde{A}| + \frac{19}{7}|\widetilde{B}| - 5, \tag{21}$$

which implies

$$|A+B| \leq |\widetilde{A}| + 3|\widetilde{B}| - 6 \quad \text{ and } \quad |A+B| \leq 2|\widetilde{A}| + 2|\widetilde{B}| - 6 - \delta(\widetilde{A}, \widetilde{B}) \tag{22}$$

except when $\delta(A,B) = \delta(B,A) = 1$ with $|A| = |B| \leq 3$. However, in view $\langle A+B-A-B\rangle = \mathbb{Z}^2$, this is only possible if we can translate A and B so that $A = B$, $|A| = |B| = 3$ and $|A+B| = 3+2+1 = 6 = |A| + 2|B| - 2 - \delta(A,B) = |A| + |B| + \frac{|A|+|B|}{2} - 3$, implying $r = r' = 0$. In such case, the theorem holds using $A = B = P = P_A = P_B$. As a result, we can assume (22) holds. Hence we can apply Theorem 1.2 to conclude that

$$|P_A \setminus A| \leq r \quad \text{ and } \quad |P_B \setminus B| \leq r. \tag{23}$$

In particular, $r \geq 0$. We must show $|P_A \setminus A| + |P_B \setminus B| \leq 2r'$ and

$$|P \setminus (x+A)| + |P \setminus (y+B)| \leq 2r + 2 + \Big||P_A| - |P_B|\Big| - \Big||A| - |B|\Big| \tag{24}$$

for some x, $y \in \mathbb{Z}^2$ with $|P \setminus (x+A)| + |P \setminus (y+B)| \leq 2r'$ unless either $P_B \subseteq P_A$ or $P_A \subseteq P_B$.

Observe that $|A| + |P_A \setminus A| = |P_A|$ and $|B| + |P_B \setminus B| = |P_B|$. Thus, if $|P_A| \geq |P_B|$, then

$$\Big||P_A| - |P_B|\Big| - \Big||A| - |B|\Big| = |P_A \setminus A| - |P_B \setminus B| - 2(|B| - \min\{|A|,\, |B|\}),$$

which is at most r in view of (23). Likewise, if $|P_B| \geq |P_A|$, then $\Big||P_A| - |P_B|\Big| - \Big||A| - |B|\Big| = |P_B \setminus B| - |P_A \setminus A| - 2(|A| - \min\{|A|,\, |B|\})$, which is again at most r by (23). In either case, the bound in (24) is at most $3r+2$.

If $\delta(\widetilde{A}, \widetilde{B}) = 1$, then $|\widetilde{A}| \geq |\widetilde{B}|$ forces $x + \tilde{A} = \widetilde{B}$ for some $x \in \mathbb{Z}^2$, whence, by an appropriate translation, we can assume $A = B$. But then $r' = r$ and $P = P_A = P_B$ trivially has

$$|P \setminus A| + |P \setminus B| = |P_A \setminus A| + |P_B \setminus B| \leq 2r = 2r',$$

as desired. Therefore we can assume

$$\delta(\widetilde{A}, \widetilde{B}) = 0, \quad r = r' + \frac{1}{2}\Big||A| - |B|\Big| - 1, \quad \text{ and } \quad 2r + 2 - \Big||A| - |B|\Big| = 2r'.$$

By definition of r, $\widetilde{A}$ and $\widetilde{B}$, we have $|A+B| = |\widetilde{A}| + 2|\widetilde{B}| - 2 + r + \delta(\widetilde{A}, \widetilde{B})$. Thus (22) implies

$$r \leq |\widetilde{B}| - \delta(\widetilde{A}, \widetilde{B}) - 4 = |\widetilde{B}| - 4 = \min\{|A|,\, |B|\} - 4. \tag{25}$$

We trivially have $|P \setminus (x+A)| \geq |P_A \setminus A|$ and $|P \setminus (y+B)| \geq |P_B \setminus B|$. Thus the inequality $|P \setminus (x+A)| + |P \setminus (y+B)| \leq 2r + 2 - \Big||A| - |B|\Big| = 2r'$ implies $|P_A \setminus A| + |P_B \setminus B| \leq 2r'$, meaning we only need to separately verify $|P_A \setminus A| + |P_B \setminus B| \leq 2r'$ when the former bound fails.

To simplify notation, whenever we consider a subset X contained in a horizontal line, we will use notation and language from $\mathbb{Z}$, such as $\operatorname{diam} X$, $<$, $\gcd(X)$, an interval in X, etc., to refer to the corresponding concept for $\pi_1(X)$. For example, an interval $I \subseteq X$ is a set such that $\pi_1(I)$ is an interval in $\mathbb{Z}e_1$, $\operatorname{diam} X = \max \pi_1(X) - \min \pi_1(X)$, and $x < y$, for elements x, $y \in X$, means $\pi_1(x) < \pi_1(y)$, etc. Likewise $\max X \in X$ is the element $x \in X$ with $\pi_1(x) = \max \pi_1(X)$, and $\min X \in X$ is the element $x \in X$ with $\pi_1(x) = \min \pi_1(X)$, which both exist when X is finite and nonempty.

By exchanging the roles of A and B if need be and translating, we may w.l.o.g. assume $P_{B_0} \subseteq P_{A_0}$. Then, by appropriately translating B and possibly applying the linear transformation $(x, y) \mapsto (-x, y)$, we can further assume one of the following three cases holds:

A: $P_{B_0} \subseteq P_{A_0}$ and $P_{A_1} \subseteq P_{B_1}$,
B: $P_{B_0} \subseteq P_{A_0}$ and $P_{B_1} \subseteq P_{A_1}$,
C: $P_{B_0} \subseteq P_{A_0}$, $\min A_0 = \min B_0$, $\min A_1 < \min B_1$, and $\max A_1 < \max B_1$.

With the translation assumptions above, let

$$P = P_{A \cup B}.$$

We handle the above three cases separately.

Case C. In this case, there is a horizontal shift ψ_1 with

$$\pi_1\big(\max \psi_1(A_0)\big) = \pi_1\big(\min \psi_1(A_1)\big) \qquad \text{and} \qquad \pi_1\big(\max \psi_1(B_0)\big) < \pi_1\big(\min \psi_1(B_1)\big),$$

and also an horizontal shift ψ_2 with

$$\pi_1\big(\max \psi_2(B_1)\big) = \pi_1\big(\min \psi_2(B_0)\big) \qquad \text{and} \qquad \pi_1\big(\max \psi_2(A_1)\big) < \pi_1\big(\min \psi_2(A_0)\big).$$

Let $A' = \mathbf{C}_{e_2}(\psi_1(A))$, $B' = \mathbf{C}_{e_2}(\psi_1(B))$, $A'' = \mathbf{C}_{e_2}(\psi_2(A))$, and $B'' = \mathbf{C}_{e_2}(\psi_2(B))$. From the above, we have $B' = B'_0$, $A' = A'_0 \cup A'_1$ with $|A'_1| = 1$, $B'' = B''_0 \cup B''_1$ with $|B''_1| = 1$, and $A'' = A''_0$. Moreover, $\langle A' + B' - A' - B' \rangle = \langle A'' + B'' - A'' - B'' \rangle = \mathbb{Z}^2$ by (8), and in view of the hypotheses of Case **C**, we find that

$$|P_{A''} \setminus A''| + |P_{B'} \setminus B'| = (|P \setminus A| - 1) + (|P \setminus B| - 1). \tag{26}$$

Now (13), (9) and (22) imply $|A' + B'| \le |A + B| \le |A'| + |B'| + 2\min\{|A'|, |B'|\} - 6$. Likewise, $|A'' + B''| \le |A + B| \le |A''| + |B''| + 2\min\{|A''|, |B''|\} - 6$. Also, (13), (9) and the hypotheses of Theorem 1.3 give

$$\begin{aligned} |A' + B'| &\le |A + B| = |A| + |B| + \min\{|A|, |B|\} - 2 + r \\ &= |A'| + 2|B'| - 2 + r - (|B| - \min\{|A|, |B|\}) \end{aligned}$$

and, likewise, $|A'' + B''| \le |B''| + 2|A''| - 2 + r - (|A| - \min\{|A|, |B|\})$. But this means we can apply Theorem 1.1 to $A' + B'$ and $A'' + B''$ to conclude $|P_{B'} \setminus B'| \le r - (|B| - \min\{|A|, |B|\})$

and $|P_{A''} \setminus A''| \leq r - (|A| - \min\{|A|, |B|\})$. Combined with (26), we obtain the desired bound

$$|P \setminus A| + |P \setminus B| \leq 2r + 2 - \Big||A| - |B|\Big| = 2r'.$$

Case B. In this case, $P_B \subseteq P_A$, implying $P = P_A$, so that we trivially have $|P \setminus A| + |P \setminus B| = 2|P_A| - |A| - |B| = 2|P_A \setminus A| + |A| - |B|$. Moreover, there is a horizontal shift ψ_1 with

$$\max \psi_1(A_0) = \pi_1\big(\min \psi_1(A_1)\big) \quad \text{and} \quad \pi_1(\max \psi_1(B_0)) \leq \pi_1\big(\min \psi_1(B_1)\big),$$

and also an horizontal shift ψ_2 with

$$\pi_1(\max \psi_2(A_1)) = \pi_1\big(\min \psi_2(A_0)\big) \quad \text{and} \quad \pi_1\big(\max \psi_2(B_1)\big) \leq \pi_1\big(\min \psi_2(B_0)\big).$$

Let $A' = \mathbf{C}_{e_2}(\psi_1(A))$, $B' = \mathbf{C}_{e_2}(\psi_1(B))$, $A'' = \mathbf{C}_{e_2}(\psi_2(A))$, and $B'' = \mathbf{C}_{e_2}(\psi_2(B))$ as in Case **C**.

If both $\pi_1\big(\max \psi_1(B_0)\big) = \pi_1\big(\min \psi_1(B_1)\big)$ and $\pi_1\big(\max \psi_2(B_1)\big) = \pi_1\big(\min \psi_2(B_0)\big)$, then $P = P_A = P_B$ and $|A| + |P \setminus A| = |B| + |P \setminus B|$. Thus, if $|A| \geq |B|$, then (23) implies $|P \setminus A| = |P \setminus B| - (|A| - |B|) \leq r - (|A| - |B|)$. Likewise, if $|B| \geq |A|$, then (23) trivially gives $|P \setminus B| = |P \setminus A| - (|B| - |A|) \leq r - (|B| - |A|)$. In either case, we see that one of the two estimates in (23) can be improved by $\Big||A| - |B|\Big|$, yielding the desired bound

$$|P \setminus A| + |P \setminus B| \leq 2r - \Big||A| - |B|\Big| = 2r' - 2.$$

Suppose equality holds in only one of the two, say $\pi_1\big(\max \psi_2(B_1)\big) = \pi_1\big(\min \psi_2(B_0)\big)$ (the other case is nearly identical using $A'' + B''$ in place of $A' + B'$). Then $B' = B'_0$, $A' = A'_0 \cup A'_1$ with $|A'_1| = 1$, $|P_{A'} \setminus A'| = |P_A \setminus A| = |P \setminus A|$ and $|P_{B'} \setminus B'| = |P_A \setminus B| - 1 = |P \setminus B| - 1$. As in Case **C**, we can apply Theorem 1.1 to $A' + B'$ to conclude

$$|P \setminus A| = |P_{A'} \setminus A'| \leq r \quad \text{and} \quad |P \setminus B| = |P_{B'} \setminus B'| + 1 \leq r + 1.$$

Moreover,

$$|A| + |P \setminus A| = |B| + |P \setminus B|.$$

Thus, if $|A| \geq |B|$, then we have $|P \setminus A| = |P \setminus B| - (|A| - |B|) \leq r + 1 - (|A| - |B|)$, while if $|B| \geq |A|$, then we instead find $|P \setminus B| = |P \setminus A| - (|B| + |A|) \leq r - (|B| - |A|)$. In either case, one of the estimates $|P \setminus A| \leq r$ or $|P \setminus B| \leq r + 1$ can be improved by at least $\Big||A| - |B|\Big| - 1$, yielding the desired bound

$$|P \setminus A| + |P \setminus B| \leq 2r + 2 - \Big||A| - |B|\Big| = 2r'.$$

It remains to consider the case when $\pi_1\big(\max \psi_2(B_1)\big) < \pi_1\big(\min \psi_2(B_0)\big)$ and $\pi_1\big(\max \psi_1(B_0)\big) < \pi_1\big(\min \psi_1(B_1)\big)$.

In this case, $B' = B'_0$, $B'' = B''_0$, $A' = A'_0 \cup A'_1$ with $|A'_1| = 1$, $A'' = A''_0 \cup A'_1$ with $|A''_0| = 1$, and

$$|P_{B'} \setminus B'| + |P_{B''} \setminus B''| = |P \setminus B| + |P_B \setminus B| - 2. \tag{27}$$

As in Case **C**, we can apply Theorem 1.1 to $A'+B'$ and $A''+B''$ to conclude $|P_{B'}\setminus B'|\leq r-(|B|-\min\{|A|,\,|B|\})$ and $|P_{B''}\setminus B''|\leq r-(|B|-\min\{|A|,\,|B|\})$. Thus $P=P_A$ and (27) give

$$\begin{aligned}|P\setminus A|+|P\setminus B| &\leq 2r+2+|P_A\setminus A|-|P_B\setminus B|-2(|B|-\min\{|A|,\,|B|\}) \\ &= 2r+2+|P_A|-|P_B|-\Big||A|-|B|\Big|.\end{aligned} \tag{28}$$

As explained at the start of the case, we have $|P\setminus A|+|P\setminus B|=2|P_A\setminus A|+|A|-|B|\leq 2r+|A|-|B|$ (with the latter inequality in view of (23)). Thus, if $|B|\geq|A|$, then $|P\setminus A|+|P\setminus B|\leq 2r-\Big||A|-|B|\Big|=2r'-2$ follows, as desired. So instead assume $|A|>|B|$. Observe that $|P\setminus B|=|P|-|B|=|P_A\setminus A|+|A|-|B|$. Using this substitution in (28) together with $|A|>|B|$ and $P_A=P$ yields $|P_A\setminus A|+|P_B\setminus B|\leq 2r+2-(|A|-|B|)=2r'$, which together with (28) gives the desired conclusions.

Case A. Let $h_0=|P_{A_0}\setminus A_0|\leq|P_A\setminus A|$, let $h_1=|P_{B_1}\setminus B_1|\leq|P_B\setminus B|$ and let $h_0'=|P_{B_0}\setminus B_0|$. We may assume $P_{A_1}\neq P_{B_1}$, and thus $P_{A_1}\subsetneq P_{B_1}$, else Case **B** applies and the proof is complete. We may also assume $P_{B_0}\neq P_{A_0}$, and thus $P_{B_0}\subsetneq P_{A_0}$, else swapping the roles of A and B results in Case **B** applying, completing the proof as before. In particular,

$$\delta(A_0,B_0)=0 \qquad \text{and} \qquad \delta(B_1,A_1)=0. \tag{29}$$

Moreover, if $|A_0|=1$, then $|P_{A_0}|=1$, whence the case hypothesis forces $P_{B_0}=P_{A_0}$, contrary to what we just assumed. Likewise, if $|B_1|=1$, then $|P_{B_1}|=1$, whence the hypotheses of Case **A** force $P_{A_1}=P_{B_1}$, contrary to what we just assumed. Therefore we may assume

$$|A_0|\geq 2 \qquad \text{and} \qquad |B_1|\geq 2. \tag{30}$$

Since $P_{B_0}\subsetneq P_{A_0}$ and $P_{A_1}\subsetneq P_{B_1}$, we also have

$$\begin{aligned}\mathrm{diam}(A_0) &= |P_{A_0}|-1>|P_{B_0}|-1=\mathrm{diam}(B_0) \qquad \text{and} \\ \mathrm{diam}(B_1) &= |P_{B_1}|-1>|P_{A_1}|-1=\mathrm{diam}(A_1).\end{aligned} \tag{31}$$

We claim that if the following inequality holds, then the proof is complete:

$$|A+B|\geq|A|+|B|-3+|A_0|+|B_1|+h_0+h_1. \tag{$\bigstar$}$$

Indeed, since $|P|=|P_{A_0}\cup P_{B_1}|=|A_0|+h_0+|B_1|+h_1$ and $|A+B|=|A|+|B|+\min\{|A|,\,|B|\}-2+r$ (by hypothesis), ($\bigstar$) implies $|P|\leq\min\{|A|,\,|B|\}+1+r$. Thus

$$\begin{aligned}|P\setminus A|+|P\setminus B| &= 2|P|-|A|-|B| \\ &\leq 2\min\{|A|,\,|B|\}+2+2r-|A|-|B| \\ &= 2r+2-\Big||A|-|B|\Big|=2r',\end{aligned}$$

which would give the desired bound.

Recall (30) and (29). Then, in view of (31), we can apply Theorem A(ii) (with $d = 1$) to A_0+B_0 unless $\gcd(A_0-A_0+B_0-B_0) \geq 2$. If $\gcd(A_0-A_0+B_0-B_0) \geq 3$, then $\gcd(A_0-A_0) \geq 3$. If $\gcd(A_0-A_0+B_0-B_0) = 2$ and $\gcd(A_0-A_0) \leq 2$, then $\gcd(A_0-A_0) = 2$ and $2 \mid \gcd(B_0-B_0)$. If $|B_0| = 1$, then $|A_0 + B_0| \geq |A_0| = |A_0| + 2|B_0| - 2$. In all other cases, we can apply Theorem A(ii) (with $d = 1$) to $A_0 + B_0$, with the result that one of the following cases holds:

(a) $\gcd(A_0 - A_0) \geq 3$, or $\gcd(A_0 - A_0) = 2$ and $\gcd(A_1 - A_1) = 2d_0 > 0$,
(b) $|A_0 + B_0| \geq |A_0| + 2|B_0| - 2$,
(c) $|A_0 + B_0| \geq |A_0| + |B_0| + h_0 - 1$.

Likewise applying Theorem A(ii) (with $d = 1$) to $B_1 + A_1$ gives three possibilities:

(a') $\gcd(B_1 - B_1) \geq 3$, or $\gcd(B_1 - B_1) = 2$ and $\gcd(A_1 - A_1) = 2d_1 > 0$,
(b') $|B_1 + A_1| \geq |B_1| + 2|A_1| - 2$,
(c') $|B_1 + A_1| \geq |B_1| + |A_1| + h_1 - 1$.

Using symmetries, we see this gives us six subcases depending on which of the three cases holds for $A_0 + B_0$ and $A_1 + B_1$. However we first make the following observations.

Claim 1. *We cannot have both* $\gcd(B_0 - B_0) \neq 1$ *and* $\gcd(B_1 - B_1) \neq 1$*, nor* $\gcd(A_0 - A_0) \neq 1$ *and* $\gcd(A_1 - A_1) \neq 1$*.*

Proof. If $\gcd(B_0 - B_0) \neq 1$ and $\gcd(B_1 - B_1) \neq 1$, then $|P_B \setminus B| \geq |B| - 2$, contradicting (25) and (23). Likewise, $\gcd(A_0 - A_0) \neq 1$ and $\gcd(A_1 - A_1) \neq 1$ would imply $|P_A \setminus A| \geq |A| - 2$, also contradicting (25) and (23). □

Claim 2. $|(A_0 + B_1) \cup (B_0 + A_1)| \geq |A_0| + |(B_0 + A_1) \cap I|$ *for any interval* $I \subseteq e_2 + \mathbb{Z}e_1$ *with* $|I| \leq \operatorname{diam} B_1$*.*

Proof. For any $x \in A_0$ with $x + \min B_1 < \min I$, we have $x + \min B_1 \in (A_0 + B_1) \setminus I$. For any $x \in A_0$ with $x + \min B_1 \geq \min I$, we have

$$x + \max B_1 \geq \min I + \max B_1 - \min B_1 > \max I,$$

with the second inequality in view of the hypothesis $\max I - \min I + 1 = |I| \leq \operatorname{diam} B_1 = \max B_1 - \min B_1$. Thus $x + \max B_1 \in (A_0 + B_1) \setminus I$. As a result, there are at least $|A_0|$ elements contained in $(A_0 + B_1) \setminus I$, and the desired lower bound readily follows. □

Subcase 1: $\gcd(A_0 - A_0) \geq 2$ and $\gcd(B_1 - B_1) \geq 2$.

In this case, Claim 1 implies $\gcd(A_1-A_1) = 1$ and $\gcd(B_0-B_0) = 1$. Consequently, Corollary 3.1 gives $|B_1+A_1| \geq 2|B_1|+|A_1|-2$ and $|A_0+B_0| \geq 2|A_0|+|B_0|-2$. Combing these estimates with Theorem C applied to A_1+B_0, we discover that $|A+B| \geq |A_0+B_0|+|B_1+A_1|+|A_1+B_0| \geq 2|A| + 2|B| - 5$, contradicting (22).

Subcase 2: (b) holds for $A_0 + B_0$ and (b') holds for $A_1 + B_1$.

In this case, the estimates given by (b) and (b') combined with Theorem C applied to A_0+B_1 imply $|A + B| \geq |A_0 + B_0| + |B_1 + A_1| + |A_0 + B_1| \geq 2|A| + 2|B| - 5$, contradicting (22).

Subcase 3: (c) holds for $A_0 + B_0$ and (c') holds for $A_1 + B_1$.

In this case, combining the estimates from (c) and (c') with Theorem C applied to $A_0 + B_1$ yields $|A + B| \geq |A_0 + B_0| + |B_1 + A_1| + |A_0 + B_1| \geq |A| + |B| - 3 + h_0 + |A_0| + h_1 + |B_1|$, establishing (★), thus completing the proof as already noted.

Subcase 4: (b) holds for $A_0 + B_0$ and $\gcd(B_1 - B_1) \geq 2$, or (b') holds for $A_1 + B_1$ and $\gcd(A_0 - A_0) \geq 2$.

By symmetry, we may w.l.o.g. assume the former case occurs. Thus

$$|A_0 + B_0| \geq |A_0| + 2|B_0| - 2 \tag{32}$$

and $\gcd(B_1 - B_1) \geq 2$. Hence Claim 1 implies $\gcd(B_0 - B_0) = 1$. In view of Subcase 1, we may assume $\gcd(A_0 - A_0) = 1$. Thus Corollary 3.1 gives

$$|A_0 + B_1| \geq |A_0| + 2|B_1| - 2. \tag{33}$$

Likewise, if $\gcd(A_1 - A_1) \neq 1$, then Corollary 3.1 gives $|B_0 + A_1| \geq |B_0| + 2|A_1| - 2$. On the other hand, if $\gcd(A_1 - A_1) = 1$, then Corollary 3.1 instead gives $|A_1 + B_1| \geq |A_1| + 2|B_1| - 2$. In consequence,

$$|A_1 + B_1| \geq |A_1| + 2|B_1| - 2 \quad \text{or} \tag{34}$$

$$|B_0 + A_1| \geq |B_0| + 2|A_1| - 2. \tag{35}$$

We will derive a contradiction in either case.

Suppose first that (34) holds. Then combining this estimate together with (32) and (33) yields

$$\begin{aligned} |A + B| &\geq |A_0 + B_0| + |A_1 + B_1| + |A_0 + B_1| \\ &\geq |A| + 2|B| - 4 + (|A_0| + 2|B_1| - 2). \end{aligned}$$

On the other hand, combining (32) and (34) together with Theorem C applied to $B_0 + A_1$ yields

$$\begin{aligned} |A + B| &\geq |A_0 + B_0| + |A_1 + B_1| + |B_0 + A_1| \\ &\geq |A| + 2|B| - 4 + (|A| + |B| - |A_0| - |B_1| - 1). \end{aligned}$$

Averaging these two estimates, we find that

$$|A + B| \geq |A| + 2|B| - 4 + \frac{|A| + |B| - 3}{2} + \frac{|B_1|}{2} > \frac{3}{2}|A| + \frac{5}{2}|B| - 5,$$

where the final estimate follows from (30). But this contradicts (22). So instead assume (35) holds.

Combining (35) together with (32) and Theorem C applied to $A_1 + B_1$ yields

$$|A + B| \geq 3|A| + 3|B| - 5 - 2|B_1| - 2|A_0|.$$

On the other hand, combining (32), (33) and Theorem C applied to $A_1 + B_1$ yields

$$|A + B| \geq |A| + 2|B| - 5 + |A_0| + |B_1|.$$

Averaging 1 copy of the first bound with two copies of the second, we find that

$$|A+B| \geq \frac{5|A|+7|B|}{3} - 5,$$

contrary to (22), which completes Subcase 4.

In view of the previous subcases, we can w.l.o.g. assume (c) holds for A_0+B_0 but not (b) nor $\gcd(A_0-A_0)\geq 2$. Thus

$$\gcd(A_0-A_0)=1 \quad \text{and} \quad |A_0+B_0| \leq |A_0|+2|B_0|-3. \tag{36}$$

In consequence, in view of (31) and (29), we can apply Theorem A(ii) (with $d=1$) to A_0+B_0 and use the refined machinery described in Section 2. In particular, letting $J_{A_0}\subseteq P_{A_0}$ and $J_{B_0}\subseteq P_{B_0}$ be the intervals defined there, we have

$$|A_0+B_0| \;\geq\; |A_0|+|B_0|-1+\max\{h_0+|J_{B_0}\setminus B_0|,\; h_0'+|J_{A_0}\setminus A_0|\}, \tag{37}$$

$$|J_{B_0}| \;\geq\; |B_0|-h_0+|J_{A_0}\setminus A_0|+|J_{B_0}\setminus B_0|, \quad \text{and} \tag{38}$$

$$|J_{A_0}| \;\geq\; |A_0|-h_0'+|J_{A_0}\setminus A_0|+|J_{B_0}\setminus B_0|. \tag{39}$$

We proceed with two claims before the next subcase. Let

$R_1\subseteq e_2+\mathbb{Z}e_1$ be an interval with $|R_1|=\operatorname{diam} B_1$ that maximizes $|(B_0+A_1)\cap R_1|$.

Claim 3. *We may assume* $|J_{B_0}|<\operatorname{diam} B_1=|B_1|+h_1-1$ *and* $|(B_0+A_1)\cap R_1|+|J_{B_0}\setminus B_0|<\operatorname{diam} B_1=|B_1|+h_1-1$*, else* $(\bigstar)$ *holds, completing the proof.*

Proof. If $|J_{B_0}|\geq \operatorname{diam} B_1$, then we can find an interval $J\subseteq J_{B_0}$ with $|J|=\operatorname{diam} B_1$ and $|J_{B_0}\setminus B_0|\geq |J\setminus B_0|$, in which case $R=J+\min A_1\subseteq e_2+\mathbb{Z}e_1$ is an interval with $|R|=\operatorname{diam} B_1$ and $|(B_0+A_1)\cap R|\geq |R|-|J_{B_0}\setminus B_0|=\operatorname{diam} B_1-|J_{B_0}\setminus B_0|$. On the other hand, if $|(B_0+A_1)\cap R_1|+|J_{B_0}\setminus B_0|\geq \operatorname{diam} B_1$, then the same conclusion holds for the interval $R=R_1\subseteq e_2+\mathbb{Z}e_1$. In either case, applying Claim 2 yields

$$|(A_0+B_1)\cup(B_0+A_1)|\geq |A_0|+\operatorname{diam} B_1-|J_{B_0}\setminus B_0|=|A_0|+|B_1|+h_1-1-|J_{B_0}\setminus B_0|.$$

Combining this estimate together with (37) and the estimate $|A_1+B_1|\geq |A_1|+|B_1|-1$ obtained from applying Theorem C to A_1+B_1, we find that

$$\begin{aligned}|A+B| \geq &(|A_0|+|B_0|-1+h_0+|J_{B_0}\setminus B_0|)+(|A_0|+|B_1|+h_1-1-|J_{B_0}\setminus B_0|)\\ &+(|A_1|+|B_1|-1)=|A|+|B|-3+|A_0|+|B_1|+h_0+h_1,\end{aligned}$$

yielding $(\bigstar)$. □

Claim 4. $|(A_0+B_1)\cup(B_0+A_1)|\geq |A_0|+|(B_0+A_1)\cap R_1|\geq |A_0|+|B_0|-h_0+|J_{A_0}\setminus A_0|$.

Proof. Applying Claim 2 with $I = R_1$ yields $|(A_0+B_1)\cup(B_0+A_1)| \geq |A_0|+|(B_0+A_1)\cap R_1|$. In view of Claim 3 and (38), we have $|(B_0+A_1)\cap R_1| \geq |(B_0+A_1)\cap(J_{B_0}+\min A_1)| \geq |J_{B_0}\cap B_0| = |J_{B_0}| - |J_{B_0}\setminus B_0| \geq |B_0| - h_0 + |J_{A_0}\setminus A_0|$, which combined with the previous inequality gives the desired bound. □

Subcase 5: (b') holds for A_1+B_1 with $\gcd(B_1-B_1)\leq 2$.

Since (b') holds for A_1+B_1, we have

$$|A_1+B_1| \geq 2|A_1|+|B_1|-2. \tag{40}$$

In view of (31), we have $|A_0|+h_0 > |B_0|$. Consequently, if $|A_0+B_1| \geq 2|A_0|+|B_1|-3$, then (37) and (40) imply

$$\begin{aligned} |A+B| &\geq 2|A|+|B|-6+|A_0|+h_0+|B_1| \\ &\geq 2|A|+2|B|-5, \end{aligned}$$

contradicting (22). Therefore we may assume

$$|A_0+B_1| \leq 2|A_0|+|B_1|-4. \tag{41}$$

On the other hand, if $|A_0+B_1| \geq |A_0|+|B_1|-1+h_1-|J_{B_0}\setminus B_0|$, then (37) and Theorem C applied to A_1+B_1 show that (★) holds, completing the proof. Thus, in view of (36), (41) and the subcase hypothesis $\gcd(B_1-B_1)\leq 2$, applying Theorem A(ii) (with $d=1$) to A_0+B_1 implies that

$$\operatorname{diam} A_0 > \operatorname{diam} B_1 \quad \text{and} \quad |A_0+B_1| \geq |A_0|+2|B_1|-2. \tag{42}$$

(Note $\operatorname{diam} A_0 > \operatorname{diam} B_1$ implies $\delta(A_0,B_1)=0$.)

Suppose $|A_0+B_0| \geq 2|A_0|+|B_0|-3$. Then combining this estimate with (40) and (42) yields

$$|A+B| \geq 2|A|+|B|-7+|A_0|+2|B_1|.$$

Using the estimate $|A_0|+h_0 > |B_0|$ (from (31)) in (37) and combining the resulting bound with (40) and Theorem C applied to A_1+B_0 yields

$$|A+B| \geq 3|A|+3|B|-3|A_0|-2|B_1|-3.$$

Averaging 1 copy of the above bound with 3 copies of the previous bound yields

$$|A+B| \geq \frac{9}{4}|A|+\frac{6}{4}|B|-6+|B_1| \geq |\widetilde{A}|+\frac{11}{4}|\widetilde{B}|-4,$$

where (30) was used for the final inequality, which contradicts (21). So we may instead assume $|A_0+B_0| \leq 2|A_0|+|B_0|-4$, which in view of (37) forces

$$h_0 \leq |A_0|-3.$$

Let $I \subseteq P_{A_0}$ be an interval with $|I| = \operatorname{diam} B_1$ such that $\gcd(A_0'+B_1-A_0'-B_1)=1$, where

$$A_0' = A_0\cap I.$$

To see why such an interval I exists, note that (42) implies $|P_{A_0}| = \operatorname{diam} A_0 + 1 > \operatorname{diam} B_1 + 1 > |I|$. Consequently, if $\gcd(B_1 - B_1) = 1$, then we need only choose I so that A_0' is nonempty (possible as $|I| = \operatorname{diam} B_1 \geq 1$ by (30)), while if $\gcd(B_1 - B_1) = 2$, then $|I| = \operatorname{diam} B_1 \geq 2$, so that the only way I could fail to exist would be if there were no consecutive elements in A_0. However, that would contradict $h_0 \leq |A_0| - 3$, so I exists. We have

$$|A_0'| \geq |I| - h_0 = |B_1| + h_1 - h_0 - 1. \tag{43}$$

As argued in Claim 3, for any $x \in A_0$ with $x > \max I$, we have $x + \max B_1 > \max(I + P_{B_1})$, and for any $x \in A_0$ with $x < \min I$, we have $x + \min B_1 < \min(I + P_{B_1})$. Consequently, there are at least $|A_0| - |A_0'|$ elements of $A_0 + B_1$ that lie outside the interval $I + P_{B_1}$, thus being distinct from any element of $A_0' + B_1 \subseteq I + P_{B_1}$. Hence

$$|A_0 + B_1| \geq |A_0' + B_1| + |A_0| - |A_0'|. \tag{44}$$

Since $|I| = \operatorname{diam} B_1 < \operatorname{diam} B_1 + 1 = |P_{B_1}|$, we have $\operatorname{diam} B_1 > \operatorname{diam} A_0'$, in turn implying $\delta(B_1, A_0') = 0$. Thus, since $\gcd(B_1 - B_1) \leq 2$ by subcase hypothesis, and since we also have $\gcd(A_0' + B_1 - A_0' - B_1) = 1$ as shown above, we can apply Theorem A(ii) (with $d = 1$) to $A_0' + B_1$ to conclude that either $|A_0' + B_1| \geq 2|A_0'| + |B_1| - 2$ or $|A_0' + B_1| \geq |A_0'| + |B_1| - 1 + h_1$. If the latter holds, then (44) implies $|A_0 + B_1| \geq |A_0| + |B_1| - 1 + h_1$. Combined with (37) and the bound $|A_1 + B_1| \geq |A_1| + |B_1| - 1$ (from Theorem C), we obtain the estimate $|A + B| \geq |A| + |B| - 3 + |A_0| + |B_1| + h_0 + h_1$, establishing ($\bigstar$) and completing the proof. So we may instead assume $|A_0' + B_1| \geq 2|A_0'| + |B_1| - 2$, which combined with (44) and (43) gives

$$|A_0 + B_1| \geq |A_0| + 2|B_1| - 3 + h_1 - h_0. \tag{45}$$

Combining the estimates in (37), (45) and (40) gives

$$|A + B| \geq 2|A| + |B| - 6 + 2|B_1| + h_1. \tag{46}$$

Combining the estimates in (37), (42) and (40) gives

$$|A + B| \geq 2|A| + |B| - 5 + 2|B_1| + h_0. \tag{47}$$

Combining the estimates in (37), Claim 4 and (40) gives

$$|A + B| \geq 2|A| + 2|B| - 3 - |B_1|. \tag{48}$$

From (47), we deduce that

$$h_1 \geq |B_1| - 1 + |A_1| \geq |B_1|,$$

else ($\bigstar$) holds, as desired, and applying this estimate in (46) gives

$$|A + B| \geq 2|A| + |B| - 6 + 3|B_1|. \tag{49}$$

Averaging 3 copies of the bound in (48) with 1 copy of the bound in (49), we obtain

$$|A + B| \geq 2|A| + \frac{7}{4}|B| - \frac{15}{4} \geq |\widetilde{A}| + \frac{11}{4}|\widetilde{B}| - \frac{15}{4},$$

contrary to (21), which completes Subcase 5.

Subcase 6: $\gcd(B_1 - B_1) = d \geq 2$. Moreover, $\gcd(A_1 - A_1) = 2d_1 > 0$ when $d = 2$.

(Note: the case $\gcd(B_1 - B_1) = 2$ with $|A_1| = 1$ was covered in Subcase 5 as (b') holds trivially when $|A_1| = 1$.)

In this case, B_1 is contained entirely in the $d\mathbb{Z}e_1$-coset $\min B_1 + d\mathbb{Z}e_1$. Let

$$h_1^{\mathsf{e}} = |P_{B_1} \cap (\min B_1 + d\mathbb{Z}e_1)| - |B_1|$$

and observe that

$$h_1 = (|B_1| + h_1^{\mathsf{e}} - 1)(d-1) + h_1^{\mathsf{e}} = dh_1^{\mathsf{e}} + (d-1)(|B_1| - 1). \tag{50}$$

Since $|A_0| + h_0 > |P_{B_0}| \geq |B_0| = |B| - |B_1|$ (by (31)), we have

$$h_0 \geq |B| - |A_0| - |B_1| + 1. \tag{51}$$

In view of (36), let $s \in [2, d]$ be the number of $d\mathbb{Z}e_1$-cosets that intersect A_0, let $x_1, \ldots, x_s \in A_0$ be representatives for these cosets, and partition $A_0 = \bigcup_{i=1}^{s} A_0^i$ with each $A_0^i = (x_i + d\mathbb{Z}e_1) \cap A_0$. We may w.l.o.g. assume $x_1 = \min A_0$ and that we have ordered the x_i in increasing cyclic order modulo de_1.

The quantity $h_0 = |P_{A_0} \setminus A_0|$ counts the number of holes in A_0, i.e., the number of elements $x \in P_{A_0} \setminus A_0$. We can more precisely count these holes as follows. For $i \in [1, s]$, let h_0^i be the number of $x \in P_{A_0} \setminus A_0$ with $x \in x_i + d\mathbb{Z}e_1$. We have $\max A_0 \equiv x_{s-\epsilon} \mod de_1$ for some $\epsilon \in [0, s-1]$. Let $h_0^{\mathsf{e}} = h_0^1 + \ldots + h_0^s$. Let ρ be the number of $x \in P_{A_0} \setminus A_0$ not counted by any h_0^i (so $x \notin x_i + d\mathbb{Z}e_1$ for all i) that also lie between $\max A_0$ and the largest element of P_{A_0} from $\min A_0 + d\mathbb{Z}e_1$. Note $\rho \in [0, d-s]$ with $\rho = 0$ occurring precisely when the elements $x_1, \ldots, x_{s-\epsilon}$ form an arithmetic progression with difference e_1 modulo de_1. With this notation, we now have

$$h_0 = h_0^{\mathsf{e}} + \frac{|A_0| + \mathsf{h}_0^{\mathsf{e}} + \epsilon - s}{s}(d-s) + \rho \geq \frac{|A_0| - s}{s}(d-s) + \frac{d}{s}h_0^{\mathsf{e}}, \tag{52}$$

generalizing the formula from (50). When $d = s$, the estimate in (52) is trivial without some bound for h_0^{e}, meaning we will need a better way to deal with estimating h_0^{e} when $s = d$. One way will be through the following setup.

For $i \in [1, s]$, we can apply Theorem A(i) (with d as defined above) and Theorem C to $A_0^i + B_1$ to conclude

$$|A_0^i + B_1| \geq |A_0^i| + |B_1| - 1 + \Omega_i, \quad \text{ where } \Omega_i = \max\{0,\ \min\{h_1^{\mathsf{e}},\ |B_1| - 2,\ |A_0^i| - 2\}\}.$$

Thus we obtain

$$|A_0 + B_1| = \sum_{i=1}^{s} |A_0^i + B_1| \geq |A_0| + s(|B_1| - 1) + \sum_{i=1}^{s} \Omega_i \geq |A_0| + s(|B_1| - 1). \tag{53}$$

Let $\Theta \subseteq [1, s]$ be a subset of indices $\alpha \in [1, s]$ with $|A_0^\alpha| - 2 < \min\{h_1^{\mathsf{e}},\ |B_1| - 2\}$ for $\alpha \in \Theta$. Note this ensures $\Omega_\alpha \geq |A_0^\alpha| - 2$ for $\alpha \in \Theta$.

Each $P_{A_0}\cap(x_i+d\mathbb{Z}e_1)$ for $i\in[1,s-\epsilon]$ has size $\frac{|A_0|+\mathsf{h}_0^{\mathsf{e}}+\epsilon}{s}$, while each $P_{A_0}\cap(x_i+d\mathbb{Z}e_1)$ for $i\in[s-\epsilon+1,s]$ has size $\frac{|A_0|+\mathsf{h}_0^{\mathsf{e}}+\epsilon}{s}-1$. Thus $h_0^i=\frac{|A_0|+h_0^{\mathsf{e}}+\epsilon}{s}-|A_0^i|$ for $i\in[1,s-\epsilon]$, and $h_0^i=\frac{|A_0|+h_0^{\mathsf{e}}+\epsilon}{s}-|A_0^i|-1$ for the ϵ indices $i\in[s-\epsilon+1,s]$. Moreover, if $\Theta\subseteq[1,s]$ is proper, then $\lceil\frac{|A_0|-\sum_{\alpha\in\Theta}|A_0^\alpha|}{s-|\Theta|}\rceil$ is a trivial lower bound for $\frac{|A_0|+\mathsf{h}_0^{\mathsf{e}}+\epsilon}{s}$. In consequence, assuming $|\Theta|<s$, we have

$$h_0^{\mathsf{e}}+|\Theta|\geq\sum_{\alpha\in\Theta}h_0^\alpha+\min\{\epsilon,\,|\Theta|\}\geq\frac{|A_0|-\sum_{\alpha\in\Theta}|A_0^\alpha|}{s-|\Theta|}|\Theta|-\sum_{\alpha\in\Theta}|A_0^\alpha|=\frac{|\Theta|}{s-|\Theta|}|A_0|-\frac{s}{s-|\Theta|}\sum_{\alpha\in\Theta}|A_0^\alpha|. \tag{54}$$

Since $\sum\limits_{\alpha\in\Theta}\Omega_\alpha\geq\sum\limits_{\alpha\in\Theta}(|A_0^\alpha|-2)$ and $|A_0^\alpha|-2<\min\{h_1^{\mathsf{e}},\,|B_1|-2\}$ for $\alpha\in\Theta$, we can combine these estimates with (54) to obtain

$$h_0^{\mathsf{e}}+\sum_{\alpha\in\Theta}\Omega_\alpha\geq\frac{|\Theta|}{s-\Theta}\Big(|A_0|-\sum_{\alpha\in\Theta}|A_0^\alpha|\Big)-3|\Theta|\geq\frac{|\Theta|}{s-|\Theta|}\Big(|A_0|-|\Theta|(|B_1|-1)\Big)-3|\Theta|, \tag{55}$$

$$h_0^{\mathsf{e}}+\sum_{\alpha\in\Theta}\Omega_\alpha\geq\frac{|\Theta|}{s-\Theta}\Big(|A_0|-\sum_{\alpha\in\Theta}|A_0^\alpha|\Big)-3|\Theta|\geq\frac{|\Theta|}{s-|\Theta|}\Big(|A_0|-|\Theta|(h_1^{\mathsf{e}}+1)\Big)-3|\Theta|. \tag{56}$$

If $d=2$, then the subcase hypotheses ensures $\gcd(B_1-B_1)=2$ and $\gcd(A_1-A_1)=2d_1>0$, whence Claim 1 implies $\gcd(A_0-A_0)=\gcd(B_0-B_0)=1$. Thus, Corollary 3.1 applied to B_0+A_1 gives

$$|B_0+A_1|\geq|B_0|+2|A_1|-2\quad\text{ when } d=2. \tag{57}$$

Moreover, in view of Subcase 5, we can assume (b') does not hold when $d=2$, in which case $|B_1+A_1|\leq|B_1|+2|A_1|-3$. Consequently, we can apply Theorem A(ii) (with $d=2$) to B_1+A_1 to conclude (in view of (29) and (31)) that

$$|B_1|+2|A_1|-3\geq|A_1+B_1|\geq|A_1|+|B_1|-1+h_1^{\mathsf{e}}\quad\text{ when } d=2. \tag{58}$$

If $\Omega_\alpha\geq h_1^{\mathsf{e}}$ for some $\alpha\in[1,2]$, then combining (37), (53) and (58) yields $|A+B|\geq|A|+|B|-3+|A_0|+|B_1|+h_0+(|B_1|-1+2h_1^{\mathsf{e}})$. Hence ($\bigstar$) holds in view of (50), as desired. So we may assume

$$\Omega_\alpha<h_1^{\mathsf{e}}\quad\text{ for all } \alpha\in[1,2] \text{ when } d=2. \tag{59}$$

We continue with the following claim.

Claim 5. $|A_1+B_1|\leq|A_1|+2|B_1|-3$, *else* ($\bigstar$) *holds, as desired.*

Proof. Assume to the contrary that

$$|A_1+B_1|\geq|A_1|+2|B_1|-2. \tag{60}$$

From (37), (60) and $|B_0+A_1|\geq|B_0|+|A_1|-1$ (by Theorem C), we have

$$|A+B|\geq2|A|+2|B|-4-|A_0|+h_0. \tag{61}$$

From (37), Claim 4 and (60), we have

$$|A+B| \geq |A| + 2|B| - 3 + |A_0|. \tag{62}$$

Suppose $d = 2$, so $s = d = 2$. Then combining (37), (51), (57), and (60) yields

$$|A+B| \geq 3|A| + 3|B| - 4 - 3|A_0| - |B_1|. \tag{63}$$

If $\Omega_\alpha \geq |A_0^\alpha| - 2$ for all $\alpha \in [1,2]$, then $\sum_{i=1}^{2} \Omega_i \geq |A_0| - 4$, whence (53) implies $|A_0 + B_1| \geq 2|A_0| + 2|B_1| - 6$. Combining this estimate with (37), (51) and (60) yields $|A+B| \geq |A| + 2|B| - 8 + |A_0| + 2|B_1|$. Averaging 1 copy of this bound with 2 copies of (63) and 5 copies of (62) yields

$$|A+B| \geq \frac{12|A| + 18|B| - 31}{8} \geq |\widetilde{A}| + \frac{11}{4}|\widetilde{B}| - \frac{31}{8},$$

contrary to (21). Therefore, in view of (59), we may assume $\Omega_{\gamma'} \geq |B_1| - 2$ for some $\gamma' \in [1,2]$. Let γ be the other index from $[1,2]$. If $\Omega_\gamma < |B_1| - 2$, then in view of (59), we have $|A_0^\gamma| - 2 \leq \Omega_\gamma < \min\{h_1^{\mathsf{e}},\ |B_1| - 2\}$, allowing us to use (54)–(56) with $\Theta = \{\gamma\}$. In such case, the estimates (37), (53), $\Omega_{\gamma'} \geq |B_1| - 2$, $h_0 \geq h_0^{\mathsf{e}}$ and (55) give

$$\begin{aligned} |A_0 + B_0| + |A_0 + B_1| &\geq (|A_0| + |B_0| - 1 + h_0) + (|A_0| + 2|B_1| - 2 + \Omega_\gamma + \Omega_{\gamma'}) \\ &\geq |B| + 2|A_0| + |B_1| - 3 + h_0^{\mathsf{e}} + \Omega_\gamma + (|B_1| - 2) \\ &\geq |B| + 3|A_0| + |B_1| - 7. \end{aligned}$$

Combining this estimate with (60) yields $|A+B| \geq |A| + |B| - 9 + 2|A_0| + 3|B_1|$. Averaging 1 copy of this bound with 3 copies of (63) and 7 copies of (62) yields

$$|A+B| \geq \frac{17|A| + 24|B| - 42}{11} \geq |\widetilde{A}| + \frac{30}{11}|\widetilde{B}| - \frac{42}{11},$$

contrary to (21). Therefore we instead conclude that $\Omega_\gamma \geq |B_1| - 2$. Since we already showed $\Omega_{\gamma'} \geq |B_1| - 2$, (53) implies that $|A_0 + B_1| \geq |A_0| + 4|B_1| - 6$. Combining this estimate with (37) and (58) yields

$$|A+B| \geq |A| + |B| - 3 + |A_0| + |B_1| + h_0 + (3|B_1| - 5 + h_1^{\mathsf{e}}).$$

Thus we may assume $3|B_1| - 5 + h_1^{\mathsf{e}} \leq h_1 - 1 = 2h_1^{\mathsf{e}} + |B_1| - 2$, with the latter inequality in view of (50), else (★) holds, as desired. Hence $h_1^{\mathsf{e}} \geq 2|B_1| - 3$. Combining (37), (51), (57), (58) and $h_1^{\mathsf{e}} \geq 2|B_1| - 3$ yields

$$|A+B| \geq 3|A| + 3|B| - 6 - 3|A_0|. \tag{64}$$

From (37), Claim 4, (58), $h_1^{\mathsf{e}} \geq 2|B_1| - 3$ and (30), we have

$$|A+B| \geq |A| + 2|B| - 5 + |A_0| + |B_1| \geq |A| + 2|B| - 3 + |A_0|. \tag{65}$$

Averaging 3 copies of (65) with 1 copy of (64) yields

$$|A+B| \geq \frac{6|A| + 9|B| - 15}{4} \geq |\widetilde{A}| + \frac{11}{4}|\widetilde{B}| - \frac{15}{4},$$

contrary to (21), which completes the case when $d = 2$. So we may now assume $d \geq 3$.

We next show that

$$|(A_0 + B_1) \cup (B_0 + A_1)| < |A_0| + |B_0|. \tag{66}$$

Suppose (66) fails. Then combining the resulting bound with (37) and (60) yields

$$|A + B| \geq |A| + 2|B| - 3 + |A_0| + h_0. \tag{67}$$

If $|A_0 + B_1| \geq |A_0| + 3|B_1| - 3$, then (37), (60) and (51) give $|A + B| \geq |A| + 2|B| - 5 + 3|B_1|$. Averaging this estimate with 3 copies of the bound from (67) (using (51) to estimate h_0) yields

$$|A + B| \geq |A| + \frac{11}{4}|B| - \frac{11}{4},$$

contrary to (21). Therefore, instead assume $|A_0 + B_1| \leq |A_0| + 3|B_1| - 4$, whence (53) implies that $s = 2$ with $\Omega_\alpha < |B_1| - 2$ for all α. Thus (52) implies $h_0 \geq \frac{1}{2}|A_0| - 1$ in view of $d \geq 3$. But then, using this estimate in (67) and (61) and averaging the resulting bound from (67) with 3 copies of the resulting bound from (61) yields

$$|A + B| \geq \frac{7|A| + 8|B| - 19}{4} \geq |\widetilde{A}| + \frac{11}{4}|\widetilde{B}| - \frac{19}{4},$$

contrary to (21). This shows that (66) holds.

Next, we show that

$$\operatorname{diam} B_0 \geq \operatorname{diam} B_1 > \operatorname{diam} A_1. \tag{68}$$

The latter inequality follows from (31). As a result, if (68) fails, then $|P_{B_0}| - 1 = \operatorname{diam} B_0 < \operatorname{diam} B_1$, whence we can apply Claim 2 with $I = P_{B_0} + \min A_1$ to obtain $|(A_0+B_1) \cup (B_0+A_1)| \geq |A_0| + |B_0|$, contradicting (66). This establishes (68), in turn implying $\delta(B_0, A_1) = 0$.

In view of Claim 1 and $\gcd(B_1 - B_1) = d \geq 3$, we have $\gcd(B_0 - B_0) = 1$. We claim that

$$|B_0 + A_1| \geq |B_0| + 2|A_1| - 2 = 2|A| + |B| - 2 - 2|A_0| - |B_1|. \tag{69}$$

Indeed, if this fails, then (68) and $\gcd(B_0 - B_0) = 1$ allow us to apply A(ii) (with $d = 1$) to $B_0 + A_1$ to conclude that there is an interval $R \subseteq B_0 + A_1 \subseteq P_{B_0} + P_{A_1} \subseteq P_{A_0} + P_{B_1}$ with $|R| \geq |B_0| + |A_1| - 1$. If $|R| \geq \operatorname{diam} B_1$, then we can find an interval $I \subseteq R \subseteq B_0 + A_1$ with $|I| = \operatorname{diam} B_1$. Applying Claim 2 then yields $|(A_0+B_1) \cup (B_0+A_1)| \geq |A_0| + |I| = |A_0| + |B_1| + h_1 - 1$, which combined with (37) and $|A_1 + B_1| \geq |A_1| + |B_1| - 1$ (from Theorem C) yields ($\bigstar$), as desired. On the other hand, if $|R| < \operatorname{diam} B_1$, then we can apply Claim 2 with $I = R$ to find $|(A_0 + B_1) \cup (B_0 + A_1)| \geq |A_0| + |R| \geq |A| + |B_0| - 1$. Combining this estimate with (37) and (60) then yields $|A + B| \geq 2|A| + 2|B| - 4 + h_0$, contrary to (22). This establishes (69).

From (37), (51), (53) and (60), we have

$$|A + B| \geq |A| + 2|B| - 2 + s(|B_1| - 1) + \sum_{i=1}^{s} \Omega_i \geq |A| + 2|B| - 4 + 2|B_1|. \tag{70}$$

From (37), (51), (69) and (60), we have

$$|A+B| \geq 3|A| + 3|B| - 4 - 3|A_0| - |B_1|. \tag{71}$$

We claim that

$$|A+B| < |A| + 2|B| - 13 + 5|B_1|. \tag{72}$$

Indeed, if (72) fails, then averaging 1 copy of the resulting bound with 15 copies of the bound in (62) and 5 copies of the bound in (71) yields

$$|A+B| \geq \frac{31|A| + 47|B| - 78}{21} \geq |\widetilde{A}| + \frac{19}{7}|\widetilde{B}| - \frac{26}{7},$$

contradicting (21). This establishes (72). But now (72) and (70) ensure that

$$s \leq 4.$$

Suppose $s \leq d-1$. Then (52) implies that $h_0 \geq \frac{|A_0|}{s} - 1$. Combining this estimate with (37), (69) and (60) gives

$$|A+B| \geq 3|A| + 2|B| - 6 - \frac{2s-1}{s}|A_0|. \tag{73}$$

Averaging s copies of the bound in (73) with $2s-1$ copies of the bound in (62) yields

$$|A+B| \geq \frac{(5s-1)|A| + (6s-2)|B| - 12s + 3}{3s-1} \geq |\widetilde{A}| + \frac{8s-2}{3s-1}|\widetilde{B}| - \frac{12s-3}{3s-1}.$$

However, for the values $s = 2, 3, 4$, the above bound becomes $|A+B| \geq |\widetilde{A}| + \frac{14}{5}|\widetilde{B}| - \frac{21}{5}$, $|A+B| \geq |\widetilde{A}| + \frac{22}{8}|\widetilde{B}| - \frac{33}{8}$ and $|A+B| \geq |\widetilde{A}| + \frac{30}{11}|\widetilde{B}| - \frac{45}{11}$, respectively, all of which contradict (21). So we may instead assume $s = d \geq 3$. Thus

$$s = d = 3 \quad \text{or} \quad s = d = 4.$$

From (37), (53), (60), (50) and $s = d$, we find that

$$\begin{aligned} |A+B| &\geq |A| + |B| - 3 + |A_0| + |B_1| + h_0 + s(|B_1| - 1) + \sum_{i=1}^{s} \Omega_i \qquad (74) \\ &= |A| + |B| - 3 + |A_0| + |B_1| + h_0 + h_1 + (|B_1| - 1) + \sum_{i=1}^{s} \Omega_i - s h_1^{\mathsf{e}}. \end{aligned}$$

Consequently, we can assume

$$\sum_{i=1}^{s} \Omega_i \leq s h_1^{\mathsf{e}} - |B_1|, \tag{75}$$

else (★) holds, as desired. In particular, $h_1^{\mathsf{e}} \geq \frac{1}{s}|B_1| > 0$. Also, (70) and (72) imply that $|B_1| \geq \frac{12-s}{5-s} \geq s+1 \geq 4$ for $s \in [3,4]$, whence

$$h_1^{\mathsf{e}} \geq 2 \quad \text{and} \quad |B_1| - 2 \geq 2. \tag{76}$$

Let us next show that there must be some $\gamma \in [1, s]$ with $\Omega_\gamma < |A_0^\gamma| - 2$. If this fails, then $\sum_{i=1}^{s} \Omega_i \geq |A_0| - 2s$. Using this estimate in (53) gives $|A_0 + B_1| \geq 2|A_0| + s|B_1| - 3s \geq$

$2|A_0|+|B_1|+s-4$, with the latter inequality from (76). Combining this bound with (37), (51) and (60) yields

$$|A+B|\geq |A|+2|B|-6+s+|A_0|+|B_1|.$$

However, averaging 3 copies of the above bound with 1 copy of (71) yields

$$|A+B|\geq \frac{6|A|+9|B|-(22-3s)+2|B_1|}{4}\geq |\widetilde{A}|+\frac{11}{4}|\widetilde{B}|-\frac{13}{4},$$

contrary to (21). So we may assume $\Omega_\gamma<|A_0^\gamma|-2$ for some $\gamma\in[1,s]$, in which case (76) ensures that

$$\Omega_\gamma\geq \min\{h_1^{\mathsf{e}},\,|B_1|-2\}\geq 2. \tag{77}$$

If $s=4$, then (70) and (72) ensure that $\Omega_i<|B_1|-2$ for all $i\in[1,s]$. If $\Omega_i\geq h_1^{\mathsf{e}}$ for at least two indices $i\in[1,s]$, then (75) implies that $2h_1^{\mathsf{e}}\leq \sum_{i=1}^{s}\Omega_i\leq 4h_1^{\mathsf{e}}-|B_1|$ in turn implying $|B_1|\leq 2h_1^{\mathsf{e}}\leq \sum_{i=1}^{s}\Omega_i$, whence (70) contradicts (72). So we see that there must be at least $s-1$ indices $\alpha\in[1,s]$ with $|A_0^\alpha|-2\leq\Omega_\alpha<\min\{h_1^{\mathsf{e}},\,|B_1|-2\}$, allowing us to use (55) with $\Theta=[1,s]\setminus\{\gamma\}$ when $s=4$. Let us next show that we can assume the same when $s=3$. If this fails for $s=3$, then either $\sum_{i=1}^{s}\Omega_i\geq 2|B_1|-4$ or $\sum_{i=1}^{s}\Omega_i\geq |B_1|-2+h_1^{\mathsf{e}}$ or $\sum_{i=1}^{s}\Omega_i\geq 2h_1^{\mathsf{e}}$. In the first case, (70) contradicts (72). In the second case, (75) implies $|B_1|-2+h_1^{\mathsf{e}}\leq\sum_{i=1}^{s}\Omega_i\leq 3h_1^{\mathsf{e}}-|B_1|$, in turn implying $h_1^{\mathsf{e}}\geq |B_1|-1$, whence $\sum_{i=1}^{s}\Omega_i\geq |B_1|-2+h_1^{\mathsf{e}}\geq 2|B_1|-3$, so that (70) again contradicts (72). Finally, in the third case, (75) instead implies $2h_1^{\mathsf{e}}\leq\sum_{i=1}^{s}\Omega_i\leq 3h_1^{\mathsf{e}}-|B_1|$, in turn implying $|B_1|\leq h_1^{\mathsf{e}}$, whence $\sum_{i=1}^{s}\Omega_i\geq 2h_1^{\mathsf{e}}\geq 2|B_1|$, so that (70) once more contradicts (72). Thus, for both remaining values of s, we see that we can use (55) with $|\Theta|=s-1$.

Now (74), (77) with $h_0\geq h_0^{\mathsf{e}}$ and (55) with $|\Theta|=s-1$ combine to yield

$$|A+B|\geq |A|+|B|+(s^2-6s+3)+s|A_0|-(s^2-3s)|B_1|. \tag{78}$$

Averaging 3 copies of the bound in (78) with $3s-8\geq 1$ (in view of $s\geq 3$) copies of the first bound in (70) (using the estimate $\sum_{i=1}^{s}\Omega_i\geq 0$) and s copies of the bound in (71) yields

$$\begin{aligned}|A+B| &\geq \frac{(6s-5)|A|+(9s-13)|B|-(20s-25)}{4s-5}\\ &\geq |\widetilde{A}|+\frac{11s-13}{4s-5}|\widetilde{B}|-\frac{20s-25}{4s-5}.\end{aligned}$$

For $s=3,4$, the respective bound given above is $|A+B|\geq|\widetilde{A}|+\frac{20}{7}|\widetilde{B}|-5$ and $|A+B|\geq |\widetilde{A}|+\frac{31}{11}|\widetilde{B}|-5$, both of which contradict (21), which at last completes the proof of Claim 5. $\square$

Since $\gcd(B_1 - B_1) = d \geq 2$, Claim 1 implies that $\gcd(B_0 - B_0) = 1$. In view of Claim 5 and Corollary 3.1, it follows that A_1 is contained in a single $d\mathbb{Z}e_1$-coset, i.e., $d \mid \gcd(A_1 - A_1)$. In view of $\gcd(B_0 - B_0) = 1$, let $t \in [2, d]$ be the number of $d\mathbb{Z}e_1$-cosets that intersect B_0. Repeating the setup from the beginning of Subcase 6 for B_0 instead of A_0, we find that

$$h_0' \geq \frac{|B_0| - t}{t}(d - t). \tag{79}$$

Similar arguments can be used to estimate the quantity $|J_{B_0} \setminus B_0|$, giving

$$\begin{aligned} |J_{B_0} \setminus B_0| \geq \frac{|J_{B_0}| - |J_{B_0} \setminus B_0| - t}{t}(d - t) &\geq \frac{|B_0| - h_0 + |J_{A_0} \setminus A_0| - t}{t}(d - t) \\ &\geq \frac{|B_0| - h_0 - t}{t}(d - t), \end{aligned} \tag{80}$$

where the second inequality follows from (38). Using (39) instead of (38), we can likewise estimate $|J_{A_0} \setminus A_0|$, giving us

$$\begin{aligned} |J_{A_0} \setminus A_0| \geq \frac{|J_{A_0}| - |J_{A_0} \setminus A_0| - s}{s}(d - s) &\geq \frac{|A_0| - h_0' + |J_{B_0} \setminus B_0| - s}{s}(d - s) \\ &\geq \frac{|A_0| - h_0' - s}{s}(d - s). \end{aligned} \tag{81}$$

Applying Corollary 3.1 to $B_0 + A_1$, we obtain

$$|B_0 + A_1| \geq |B_0| + t(|A_1| - 1). \tag{82}$$

Claim 6. $|A_1| + |B_1| + h_1^{\mathsf{e}} - 1 \leq |A_1 + B_1| \leq |B_1| + 2|A_1| - 3$.

Proof. When $d = 2$, the claim follows from (58). So we may assume $d \geq 3$. Since A_1 is contained in a $d\mathbb{Z}e_1$-coset with $\gcd(B_1 - B_1) = d$ and $\operatorname{diam}(B_1) > \operatorname{diam}(A_1)$ (in view of (31)), it suffices in view of (29) to show $|A_1 + B_1| \leq |B_1| + 2|A_1| - 3$ as the remaining conclusion of the claim will then follow from applying Theorem A(ii) (with $d = \gcd(B_1 - B_1)$) to $B_1 + A_1$. Assuming by contradiction that

$$|A_1 + B_1| \geq |B_1| + 2|A_1| - 2, \tag{83}$$

we can combine (83), (37), and Claim 4 to yield

$$|A + B| \geq 2|A| + 2|B| - 3 - |B_1|. \tag{84}$$

Combining (83), (37), and (53) (using the estimates $h_0 \geq 0$) yields

$$|A + B| \geq 2|A| + |B| - (s + 3) + s|B_1|. \tag{85}$$

Averaging s copies of the bound in (84) with 1 copy of the bound from (85) gives us

$$|A + B| \geq \frac{(2s + 2)|A| + (2s + 1)|B| - (4s + 3)}{s + 1} \geq |\widetilde{A}| + \frac{3s + 2}{s + 1}|\widetilde{B}| - \frac{4s + 3}{s + 1}.$$

For $s \geq 3$, the above bounds yields $|A + B| \geq |\widetilde{A}| + \frac{11}{4}|\widetilde{B}| - \frac{15}{4}$, contrary to (21). It remains to handle the case $s = 2$.

In this case, (52) implies $h_0 \geq \frac{1}{2}|A_0| - 1$ in view of $d \geq 3$. Using this estimate together with (83), (37), and (53) (and recalling that $s = 2$) yields

$$|A+B| \geq 2|A| + |B| - 6 + \frac{1}{2}|A_0| + 2|B_1|. \tag{86}$$

From (37), (51), (83) and (82) (using $t \geq 2$), we obtain

$$|A+B| \geq 4|A| + 3|B| - 4 - 4|A_0| - 2|B_1|. \tag{87}$$

Averaging 14 copies of (84) with 8 copies of (86) and 1 copy of (87) yields

$$|A+B| \geq \frac{48|A| + 39|B| - 94}{23} \geq |\widetilde{A}| + \frac{64}{23}|\widetilde{B}| - \frac{94}{23},$$

contrary to (21). This completes Claim 6. □

From (37), Claim 6 and Claim 4, we obtain

$$|A+B| \geq |A| + |B| - 2 + |A_0| + |B_0| + \Big(|J_{B_0} \setminus B_0| + |J_{A_0} \setminus A_0| + h_1^{\mathsf{e}}\Big). \tag{88}$$

From (37), Claim 6 and the first estimate in (53), we obtain

$$|A+B| \geq |A| + (s+1)|B| - (s+2) + |A_0| - s|B_0| + \Big(h_0 + |J_{B_0} \setminus B_0| + h_1^{\mathsf{e}} + \sum_{i=1}^{s}\Omega_i\Big), \tag{89}$$

$$|A+B| \geq |A| + (s+1)|B| - (s+2) + |A_0| - s|B_0| + \Big(h_0' + |J_{A_0} \setminus A_0| + h_1^{\mathsf{e}} + \sum_{i=1}^{s}\Omega_i\Big). \tag{90}$$

From (37), Claim 6 and (82), we obtain

$$|A+B| \geq (t+1)|A| + |B| - (t+2) - t|A_0| + |B_0| + \Big(h_0 + |J_{B_0} \setminus B_0| + h_1^{\mathsf{e}}\Big), \tag{91}$$

$$|A+B| \geq (t+1)|A| + |B| - (t+2) - t|A_0| + |B_0| + \Big(h_0' + |J_{A_0} \setminus A_0| + h_1^{\mathsf{e}}\Big). \tag{92}$$

Let $r = \min\{s,\, t\} \in [2, d]$. Since (89)–(92) are clearly monotonically increasing with s and t, these bounds all hold replacing s or t by r.

Now suppose that the following inequalities hold, where $x \in [0,1]$ and $\epsilon \geq 0$ are as yet unspecified real numbers:

$$|A+B| \geq |A| + |B| - 2 + |A_0| + |B_0|, \tag{93}$$

$$|A+B| \geq |A| + (r+1)|B| - (r+2) + |A_0| - r|B_0| + x\min\{|A_0|,\, |B_0|\} - \epsilon, \tag{94}$$

$$|A+B| \geq (r+1)|A| + |B| - (r+2) - r|A_0| + |B_0| + x\min\{|A_0|,\, |B_0|\} - \epsilon. \tag{95}$$

Observing that the system of inequalities given by (93)–(95) is completely symmetric with respect to the variables $|A_0|$ and $|B_0|$, it does not matter whether $|A_0|$ or $|B_0|$ attains the minimum in $\min\{|A_0|,\, |B_0|\}$, and we will w.l.o.g. assume $|A_0| = \min\{|A_0|,\, |B_0|\}$ for the following calculation (the case $|B_0| = \min\{|A_0|,\, |B_0|\}$ being nearly identical): averaging $r^2 - 1 - x(r+1)$

copies of of the bound in (93) with $r+1-x$ copies of the bound in (94) and $r+1+x$ copies of the bound in (95), we obtain

$$\begin{aligned} |A+B| &\geq \frac{\big((2r+1)(r+1)-x\big)|A|+(2r+1)(r+1-x)|B|-2(r+1)(2r+1-x+\epsilon)}{(r+1)(r+1-x)} \\ &\geq |\widetilde{A}|+\frac{3r+1-x}{r+1-x}|\widetilde{B}|-\frac{2(2r+1-x+\epsilon)}{r+1-x}. \end{aligned} \tag{96}$$

Moreover, if $x<1$ and any of the bounds in (93)–(95) is strict, then the bound in (96) will also be strict. The derivative with respect to r of the above expression is $\frac{2(\epsilon+(1-x)(|\widetilde{B}|-1))}{(r+1-x)^2}$, which is non-negative for $x\in[0,1]$, meaning the bound in (96) is minimized for small r. In view of (88), (89) and (91), the bounds (93)–(95) hold with $x=\epsilon=0$. Consequently, if $r\geq 6$, then (96) with $x=\epsilon=0$ gives

$$|A+B|\geq |\widetilde{A}|+\frac{19}{7}|\widetilde{B}|-\frac{26}{7},$$

contrary to (21), meaning we may now assume

$$r\leq 5.$$

Suppose $d\geq r+2$. Then (79) and (52) imply $h_0'\geq \frac{2}{t}|B_0|-2\geq (1-\frac{t}{6})|B_0|-\frac{3}{2}$ or $h_0\geq \frac{2}{s}|A_0|-2\geq (1-\frac{s}{6})|A_0|-\frac{3}{2}$ (depending on whether $r=t$ or $r=s$), with the latter inequalities in view of $|A_0|\geq s$ and $|B_0|\geq t$ (which hold trivially in view of the definitions of s and t). Moreover, if $r=2$, then we can slightly improve this to $h_0'>(1-\frac{t}{6})|B_0|-\frac{3}{2}$ or $h_0>(1-\frac{s}{6})|A_0|-\frac{3}{2}$. As a result, (88)–(92) imply that (93)–(95) hold with $x=1-\frac{r}{6}$ and $\epsilon=\frac{3}{2}$, and in the case $r=2$, both (94) and (95) hold strictly. Thus (96) yields

$$|A+B|\geq |\widetilde{A}|+\frac{19}{7}|\widetilde{B}|-\frac{\frac{13}{3}r+3}{\frac{7}{6}r},$$

with a strict inequality for $r=2$. For $r=2$, this gives $|A+B|>|\widetilde{A}|+\frac{19}{7}|\widetilde{B}|-5$, and for $r\geq 3$ we instead have $|A+B|\geq|\widetilde{A}|+\frac{19}{7}|\widetilde{B}|-\frac{32}{7}$, both of which contradict (21). It remains to consider the cases when $s,t\in[d-1,d]$ with $r=\min\{s,\,t\}\leq 5$.

Suppose $s=t=d-1$. Then (52) implies that $h_0\geq\frac{1}{s}|A_0|-1$ and (80) implies $|J_{B_0}\setminus B_0|\geq \frac{1}{s}|B_0|-\frac{1}{s}h_0-1$, implying $h_0+|J_{B_0}\setminus B_0|\geq\frac{1}{s}|B_0|+\frac{s-1}{s^2}|A_0|-\frac{2s-1}{s}$. We can apply this inequality to estimate the parenthetical quantities in (89) and (91) for each of the values $s=r\in[2,5]$ individually, and estimate the parenthetical quantity in (88) by 0. Then, for $s=2$, averaging 3 copies of (88) with 13 copies of the resulting bound in (89) and 11 copies of the resulting bound in (91) yields $|A+B|\geq\frac{49|A|+53|B|-138}{27}\geq|\widetilde{A}|+\frac{25}{9}|\widetilde{B}|-\frac{46}{9}$, contrary to (21) in view of $|\widetilde{B}|\geq 2$. For $s=3$, averaging 156 copies of (88) with 111 copies of the resulting bound in (89) and 105 copies of the resulting bound in (91) yields $|A+B|\geq\frac{687|A|+705|B|-1752}{372}$, contrary to (21). For $s=4$, averaging 205 copies of (88) with 81 copies of the resulting bound in (89) and 79 copies of the resulting bound in (91) yields $|A+B|\geq\frac{681|A|+689|B|-1650}{365}$, contrary to (21). For $s=5$, averaging 546 copies of (88) with 151 copies of the resulting bound in (89) and 149 copies of the resulting bound in (91) yields $|A+B|\geq \frac{1591|A|+1601|B|-3732}{846}$, contrary to (21).

Suppose $s=r=d-1$ and $t=d$. Then (81) implies that $|J_{A_0}\setminus A_0|\geq \frac{1}{s}|A_0|-\frac{1}{s}h'_0-1$ with $t=s+1$. We can apply this inequality to estimate the parenthetical quantities in (88), (90) and (92), yielding

$$|A+B|\geq |A|+|B|-3+\frac{s+1}{s}|A_0|+|B_0|-\frac{1}{s}h'_0, \tag{97}$$

$$|A+B|\geq |A|+(s+1)|B|-(s+3)+\frac{s+1}{s}|A_0|-s|B_0|+\frac{s-1}{s}h'_0, \tag{98}$$

$$|A+B|\geq (s+2)|A|+|B|-(s+4)-\frac{s^2+s-1}{s}|A_0|+|B_0|+\frac{s-1}{s}h'_0. \tag{99}$$

Averaging s^3+s^2-2s-1 copies of the bound in (97) with s^2+2s copies of the bound in (98) and s^2+2s+1 copies of the bound in (99) yields

$$|A+B|\geq \frac{(2s^3+6s^2+5s+1)|A|+(2s^3+5s^2+2s)|B|-(5s^3+14s^2+9s+1)+(s^2+s-1)h'_0}{s(s+1)(s+2)}.$$

For the values $s=2,3,4,5$, the above bound, using $h'_0\geq 0$, becomes $|A+B|\geq \frac{51|A|+40|B|-115}{24}$, $|A+B|\geq \frac{124|A|+105|B|-289}{60}$, $|A+B|\geq \frac{245|A|+216|B|-581}{120}$, and $|A+B|\geq \frac{426|A|+385|B|-1021}{210}$, respectively, all of which contradict (21).

If $s=d$ and $t=r=d-1$, then (80) implies $|J_{B_0}\setminus B_0|\geq \frac{1}{t}|B_0|-\frac{1}{t}h_0-1$. In view of the symmetry between the variables $|A_0|$ and $|B_0|$ in (88)–(92), we can then repeat the above calculation, swapping the roles of $|A_0|$ and $|B_0|$, of s and t and of h'_0 and h_0, and using (89) and (91) in place of (90) and (92), to yield the same contradiction as in the last paragraph. So it remains to consider the case when $r=s=t=d\in[2,5]$.

If $\Omega_i\geq h_1^{\mathsf{e}}$ for at least $s-1$ indices $i\in[1,s]$, then (89), $s=d$ and (50) imply that ($\bigstar$) holds, as desired. Thus

$$h_1^{\mathsf{e}}>0 \tag{100}$$

and, since Claims 5 and 6 imply that $h_1^{\mathsf{e}}\leq |B_1|-2$, we conclude that there must be distinct indices $\alpha,\beta\in[1,s]$ with $|A_0^{\alpha}|-2\leq \Omega_\alpha<h_1^{\mathsf{e}}\leq |B_1|-2$ and $|A_0^{\beta}|-2\leq \Omega_\beta<h_1^{\mathsf{e}}\leq |B_1|-2$. Consequently, letting $\Theta=\{\alpha,\beta\}$, we can apply (56) with $|\Theta|=2$ for $s\geq 3$. Before using this estimate, let us first show that we must have $\Omega_\gamma<|A_0^{\gamma}|-2$ for some $\gamma\in[1,s]$.

If $\Omega_i\geq |A_0^i|-2$ for every $i\in[1,s]$, then $\sum\limits_{i=1}^{s}\Omega_i\geq |A_0|-2s$. Using this estimate along with (51) and (100) in (89) yields

$$|A+B|\geq |A|+(s+1)|B|-3s+|A_0|-(s-1)|B_0|. \tag{101}$$

Using the estimates (51), (100) and $t=s$ in (91) yields

$$|A+B|\geq (s+1)|A|+|B|-s-(s+1)|A_0|+2|B_0|. \tag{102}$$

Averaging s^2-3 copies of (88) (using (100) to estimate the quantity in parenthesis) with $s+3$ copies of (101) and s copies of (102) yields

$$|A+B| \geq \frac{(2s+2)|A|+(2s+5)|B|-(5s+9)+\frac{3}{s}}{s+2} > |\widetilde{A}| + \frac{(3s+5)|\widetilde{B}|-(5s+9)}{s+2}.$$

The derivative with respect to s of the above expression is $\frac{|\widetilde{B}|-1}{(s+2)^2} \geq 0$, meaning the bound will be minimized for the minimal value of $s=d\geq 2$, implying $|A+B| > |\widetilde{A}| + \frac{11}{4}|\widetilde{B}| - \frac{19}{4}$, contrary to (21). So we may now assume there exists some $\gamma \in [1,s]$ with $\Omega_\gamma < |A_0^\gamma| - 2$.

From (88), we have

$$|A+B| \geq |A|+|B|-2+|A_0|+|B_0|+h_1^{\mathsf{e}}. \tag{103}$$

Observe that $\gamma\in[1,s]$ must be distinct from the distinct indices α, $\beta \in [1,s]$ with $\Omega_\gamma \geq h_1^{\mathsf{e}}$ (as $h_1^{\mathsf{e}} \leq |B_1|-2$ from Claims 5 and 6), which implies that $s\geq 3$. In view of $\Omega_\gamma \geq h_1^{\mathsf{e}}$, $h_0 \geq h_0^{\mathsf{e}}$ and and (56) (applied with $\Theta=\{\alpha,\beta\}$, so $|\Theta|=2$), we have $h_0+\sum_{i=1}^{s}\Omega_i \geq h_0^{\mathsf{e}} + (\Omega_\alpha+\Omega_\beta)+\Omega_\gamma \geq \frac{2}{s-2}(|A_0|-2h_1^{\mathsf{e}}-2)-6+h_1^{\mathsf{e}}$. Applying this estimate in (89) yields

$$|A+B| \geq |A|+(s+1)|B| - \frac{s^2+6s-12}{s-2} + \frac{s}{s-2}|A_0| - s|B_0| + \frac{2s-8}{s-2}h_1^{\mathsf{e}}. \tag{104}$$

Using the estimate (51) in (91) and recalling that $t=s$ gives

$$|A+B| \geq (s+1)|A|+|B|-(s+1)-(s+1)|A_0|+2|B_0|+h_1^{\mathsf{e}}. \tag{105}$$

Averaging $(s^2-s-4)s$ copies of the bound in (103) with $s^2+s-6=(s-2)(s+3)$ copies of the bound in (104) and $(s-1)s$ copies of the bound in (105) yields $|A+B|$ being at least

$$\frac{(2s^3-4s-6)|A|+(2s^3+2s^2-10s-6)|B|-(4s^3+7s^2-3s-36)+(s^3+2s^2-7s-24)h_1^{\mathsf{e}}}{s^3+s^2-4s-6}$$

$$\geq |\widetilde{A}| + \frac{(3s^3+s^2-10s-6)|\widetilde{B}|-(4s^3+7s^2-3s-36)+(s^3+2s^2-7s-24)h_1^{\mathsf{e}}}{s^3+s^2-4s-6}. \tag{106}$$

The coefficient of h_1^{e} in the numerator of (106) is non-negative for $s\geq 3$. Thus, using the estimate $h_1^{\mathsf{e}} \geq 1$ (from (100)), the bound in (106), for $s=4,5$, becomes $|A+B| \geq |\widetilde{A}| + \frac{162|\widetilde{B}|-276}{58}$ and $|A+B| \geq |\widetilde{A}| + \frac{344|\widetilde{B}|-508}{124}$, respectively, both of which contradict (21). It remains to consider the case $s=t=d=3$, for which (106) implies $|A+B|\geq 2|A|+2|B|-7$, which contradicts (21) unless $|A|$, $|B|\leq 7$. From (100) and Claims 6 and 5, we see that $1\leq h_1^{\mathsf{e}} \leq |B_1|-2$. Thus $|B_1|\geq 3$, whence (50) and (100) imply $|P_B\setminus B| \geq h_1 = 3h_1^{\mathsf{e}}+2(|B_1|-1)\geq 7\geq |B|$, contrary to (25) and (23), which completes the proof. □

4. Further Commentary and Lower Bound Examples

The examples below show that the bounds for $|P_A\setminus A|$, $|P_A\setminus A|+|P_B\setminus B|$ and $|P\setminus A|+|P\setminus B|$ in Theorem 1.3 are all tight. What is *not* tight in Theorem 1.3 is the hypothesis $|A+B| \leq |A|+\frac{19}{7}|B|-5$, and we would expect the theorem to remain true under a weaker small sumset hypothesis, likely replacing $\frac{19}{7}$ with something closer to 3. It is also plausible that the bound for $|P_B \setminus B|$ could be improved when $r' < r$, as the examples below showing tightness for $|P_B \setminus B| \leq r$ all have $r = r'$.

Example 1. Let $k \geq 3$ and $r \in [0, k-3]$ be integers. Let

$$A = \big([1, k-2] \times \{0\}\big) \cup \big(\{0, r+1\} \times \{1\}\big) \quad \text{and} \quad B = \big([0, k-3] \times \{0\}\big) \cup \big(\{0, r+1\} \times \{1\}\big).$$

Then $|A| = |B| = k$, $\delta(A, B) = 0$, $P_A = \big([1, k-2] \times \{0\}\big) \cup \big([0, r+1] \times \{1\}\big)$, $P_B = \big([0, k-3] \times \{0\}\big) \cup \big([0, r+1] \times \{1\}\big)$, $P = \big([0, k-2] \times \{0\}\big) \cup \big([0, r+1] \times \{1\}\big)$ and $|A+B| = |A| + 2|B| - 2 + r \leq |A| + 3|B| - 5$. Also,

$$|P_A \setminus A| = |P_B \setminus B| = r \quad \text{and} \quad |P \setminus A| + |P \setminus B| = 2r + 2 = 2r + 2 + |A| - |B|,$$

showing tightness in the bounds from Theorem 1.3.

Example 2. Let $a > b \geq 2$ be even integers and let

$$A = \big([0, a/2-1] \times \{0\}\big) \cup \big([0, a/2-1] \times \{1\}\big) \quad \text{and} \quad B = \big([0, b/2-1] \times \{0\}\big) \cup \big([0, b/2-1] \times \{1\}\big).$$

Then $|A| = a$, $|B| = b$, $\delta(A, B) = 0$, $P_A = P = A$, $P_B = B$ and $|A+B| = |A| + |B| + \frac{|A|+|B|}{2} - 1$, so $r' = 0$ and $r = \frac{|A|-|B|}{2} - 1$. However,

$$|P \setminus A| + |P \setminus B| = |A| - |B| = 2r + 2 = 2r + 2 - \Big||A| - |B|\Big| + \Big||P_A| - |P_B|\Big|,$$

showing the bound for $|P \setminus A| + |P \setminus B|$ can be tight and is not bounded as a function of r'.

Example 3. Let $a \geq b + 2 \geq 4$ and $r \in [a - b - 2, a - 3]$ be integers. Let

$$A = \big([0, a-3] \times \{0\}\big) \cup \big(\{0, r+1\} \times \{1\}\big) \quad \text{and} \quad B = \big([0, b-2] \times \{0\}\big) \cup \big(\{0\} \times \{1\}\big).$$

Then $|A| = a$, $|B| = b$, $\delta(A, B) = 0$, $P_A = \big([0, a-3] \times \{0\}\big) \cup \big([0, r+1] \times \{1\}\big)$, $P_B = \big([0, b-2] \times \{0\}\big) \cup \big(\{0\} \times \{1\}\big)$, and $P = \big([0, a-3] \times \{0\}\big) \cup \big([0, r+1] \times \{1\}\big)$. Also,

$$\begin{aligned} &2|A| + |B| - 4 \leq |A+B| = |A| + 2|B| - 2 + r \leq 2|A| + 2|B| - 5, \\ &|P_B \setminus B| = 0, \quad |P_A \setminus A| = r, \quad \text{and} \\ &|P \setminus A| + |P \setminus B| = 2r + |A| - |B| \leq 3r + 2 = 2r + 2 + |P_A| - |P_B| - \Big||A| - |B|\Big|. \end{aligned}$$

Furthermore, choosing $r = a - b - 2 = |A| - |B| - 2$, the inequality above becomes an equality, showing the bound $3r+2$ can be tight in Theorem 1.3, and $|P_B \setminus B| + |P_A \setminus A| = |P_A \setminus A| = r = 2r'$, showing this bound to be tight as well.

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University of Memphis, Department of Mathematical Sciences, Memphis, TN 38152