On the Number of m-term Zero-Sum Subsequences*

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1 Introduction

A sequence S of terms from an abelian group is *zero-sum* if the sum of the terms of S is zero. In 1961 Erdős, Ginzburg and Ziv proved that any sequence of $2m-1$ terms from an abelian group of order m contains an m-term zero-sum subsequence [10]. This sparked a flurry of generalizations, variations and extensions [1] [3] [7] [8] [11] [13] [14] [15] [16] [17] [18] [22] [26] [27] [28] [37]. Since a sequence from the cyclic group $\mathbb{Z}/m\mathbb{Z}$ consisting of only 0's and 1's has its m-term zero-sum subsequences in exact correspondence with its m-term monochromatic subsequences, then the Erdős-Ginzburg-Ziv Theorem can be viewed as a generalization of the pigeonhole principle for m pigeons and 2 boxes. In essence, the Erdős-Ginzburg-Ziv Theorem expresses the idea that often the best way to avoid zero-sums is to consider sequences with very few distinct terms.

For sequences whose length is greater than $2m-1$, a natural question to ask is *how many* m-term zero-sum subsequences can one expect. If the sequence S has length n and consists of at most two distinct terms, then there will be at least $\binom{\lceil \frac{n}{2} \rceil}{m} + \binom{\lfloor \frac{n}{2} \rfloor}{m}$ m-term monochromatic subsequences. Thus

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*AMS Subject Classification: 11B75 (11B50) if the best way to avoid m-term zero-sum subsequences were still to use only two distinct residues from $\mathbb{Z}/m\mathbb{Z}$, then one would expect there to always be at least $\binom{\lceil \frac{n}{2} \rceil}{m} + \binom{\lfloor \frac{n}{2} \rfloor}{m}$ m-term zero-sum subsequences. This was conjectured by Bialostocki in 1989 [4] and later appeared in [5].

Conjecture 1.1. *If S is a sequence of n terms from the cyclic group $\mathbb{Z}/m\mathbb{Z}$, then S has at least $\binom{\lceil \frac{n}{2} \rceil}{m} + \binom{\lfloor \frac{n}{2} \rfloor}{m}$ m-term zero-sum subsequences.*

A few years after the conjecture was made, Kisin verified Conjecture 1.1 in the case $m = p^\alpha$ and $m = p^\alpha q$, where p and q are primes and $\alpha \geq 1$, and expressed reasons why the conjecture might fail for m not of this form [31]. At the same time, Füredi and Kleitman showed that Conjecture 1.1 held for sufficiently large n (of order m^{6m}), as well as for m of the form $m = pq$, where p and q are distinct primes, and showed that $2\binom{\lfloor \frac{n}{2} \rfloor}{m} - m^2\binom{\lfloor \frac{n}{2} \rfloor - 1}{m-1}$ was a general lower bound on the number of m-term zero-sum subsequences [12]. Their results, contrary to those of Kisin, led them to strongly believe the conjecture of Bialostocki to be true for $n > 4m$. Unfortunately, the lower bound shown by Füredi and Kleitman, while being very nice asymptotically for large n and fixed m, tells us very little for small n, particularly if m is also large.

The aim of this paper is to give a proof, using some recently developed machinery from zero-sum Ramsey theory, of the following general bound on the number of m-term zero-sum subsequences.

Theorem 1.1. *If S is a sequence of n terms from an abelian group G of order $m \geq 30$, then S contains at least $\min\left\{\binom{\lceil \frac{n}{2} \rceil}{m} + \binom{\lfloor \frac{n}{2} \rfloor}{m}, \binom{n-m}{\lceil \frac{2m-1}{3} \rceil}\right\}$ m-term zero-sum subsequences.*

Unlike the general bound of Füredi and Kleitman, the bound given by Theorem 1.1 is much more accurate for sequences of small length, and, as will be shown in section 3, verifies Conjecture 1.1 for $n \leq 6\frac{1}{3}m$. Ironically, this confirms the conjecture of Bialostocki for those cases least thought to be true. Theorem 1.1 also gives a bound for more general abelian groups in addition to cyclic groups.

2 Preliminaries

Let $(G, +, 0)$ be an abelian group. If A, $B \subseteq G$, then their *sumset*, $A+B$, is the set of all possible pairwise sums, i.e. $\{a+b \mid a \in A,\, b \in B\}$. A set $A \subseteq G$ is *H_a-periodic*, if it is the union of H_a-cosets for some subgroup H_a of G (note this definition allows H_a to be trivial). We say that A is *maximally* H_a-periodic, if A is H_a-periodic, and H_a is the maximal subgroup for which A is H_a-periodic; in this case, $H_a = \{x \in G \mid x + A = A\}$, and H_a is sometimes referred to as the *stabilizer* of A. A set which is maximally H_a-periodic, with H_a the trivial group, is *aperiodic*, and otherwise we refer to A as *periodic*. An *H_a-hole* of A (where the subgroup H_a is usually understood), is an element $\alpha \in (A + H_a) \setminus A$. For notational convenience, we use $\phi_a : G \to G/H_a$ to denote the natural homomorphism. If S is a sequence of elements from G, then an *n-set partition* of S is a partition of the sequence S into n nonempty subsequences, $A_1, \ldots, A_n$, such that the terms in each subsequence A_i are all distinct (thus allowing each subsequence A_i to be considered a set). Also, $|S|$ denotes the cardinality of S, if S is a set, and the length of S, if S is a sequence. Finally, if S' is a subsequence of S, then $S \setminus S'$ denotes the subsequence of S obtained by deleting all terms in S'.

We begin by stating Kneser's Theorem [32] [29] [33] [30] [35] [24]. The case with m prime is known as the Cauchy-Davenport Theorem [9].

Kneser's Theorem. *Let G be an abelian group, and let $A_1, A_2, \ldots, A_n$ be a collection of finite, nonempty subsets of G. If $\sum_{i=1}^{n} A_i$ is maximally H_a-periodic, then*

$$\left|\sum_{i=1}^{n} \phi_a(A_i)\right| \geq \sum_{i=1}^{n} |\phi_a(A_i)| - n + 1.$$

Note that if A is maximally H_a-periodic, then $\phi_a(A)$ is aperiodic. Also, observe that if $A+B$ is maximally H_a-periodic and $\rho = |A+H_a|-|A|+|B+H_a|-|B|$ is the number of holes in A and B, then Kneser's Theorem implies $|A+B| \geq |A|+|B|-|H_a|+\rho$. Consequently, if either A or B contains a unique element from some H_a-coset, then $|A+B| \geq |A|+|B|-1$. More generally, if $\rho = \sum\limits_{i=1}^{n} |H_a+A_i|-|A_i|$ is the total number of holes in the A_i, then $\left|\sum\limits_{i=1}^{n} A_i\right| \geq \sum\limits_{i=1}^{n} |A_i|-(n-1)|H_a|+\rho$. Hence, if $\left|\sum\limits_{i=1}^{n} A_i\right| < \sum\limits_{i=1}^{n} |A_i|-n+1$, then $\rho < (n-1)(|H_a|-1)$.

The following characterizes when a sufficiently compressed n-set partition exists [20] [2].

Proposition 2.1. *Let n_1 and n_0 be positive integers with $n_0 \leq n_1$. A sequence S of terms from G has an n_1-set partition $A = A_1, \ldots, A_{n_1}$ with $|A_i| = 1$ for $i > n_0$ (and $||A_i|-|A_j|| \leq 1$ for $i, j \leq n_0$) if and only if $|S| \geq n_1$, and for every nonempty subset $X \subseteq G$ with $|X| \leq \frac{|S|-n_1-1}{n_0}+1$ there are at most $n_1+(|X|-1)n_0$ terms of S from X. In particular, S has an n_1-set partition if and only if $|S| \geq n_1$ and the multiplicity of every term of S is at most n_1.*

The next simple proposition can often be quite useful when dealing with n-set partitions [2].

Proposition 2.2. *Let S be a finite sequence of elements from an abelian group G, and let $A = A_1, \ldots, A_n$ be an n-set partition of S, where $|\sum\limits_{i=1}^{n} A_i| = r$, and $\max\limits_{i}\{|A_i|\} = s$.*

(i) There exists a subsequence S' of S and an n'-set partition $A' = A_{i_1}, \ldots, A_{i_{n'}}$ of S', which is a subsequence of the n-set partition $A = A_1, \ldots, A_n$, such that $n' \leq r-s+1$ and $|\sum\limits_{j=1}^{n'} A_{i_j}| = r$.

(ii) There exists a subsequence S' of S of length at most $n+r-1$, and an n-set partition $A' = A'_1, \ldots, A'_n$ of S', where $A'_i \subseteq A_i$ for $i = 1, \ldots, n$, such that $|\sum\limits_{i=1}^{n} A'_i| = r$.

The following theorem [20] [23] is a recent generalization of results of Mann [34], Olson [36], Bolobás and Leader [6], and Hamidoune [25].

Theorem 2.1. *Let S' be a subsequence of a finite sequence S of terms from an abelian group G,* *let $A = A_1, \ldots, A_n$ be an n-set partition of S', and let $a_i \in A_i$ for $i \in \{1, \ldots, n\}$. Then there exists an n-set partition $A' = A'_1, \ldots, A'_n$ of a subsequence S'' of S with sumset H_a-periodic, $|S'| = |S''|$, $\sum\limits_{i=1}^{n} A_i \subseteq \sum\limits_{i=1}^{n} A'_i$, $a_i \in A'_i$ for $i \in \{1, \ldots, n\}$, and*

$$\left|\sum_{i=1}^{n} A'_i\right| \geq \big(E(A', H_a) + (N(A', H_a) - 1)n + 1\big)\,|H_a|,$$

where $N(A', H_a) = \frac{1}{|H_a|}|\bigcap\limits_{i=1}^{n}(A'_i + H_a)|$ and $E(A', H_a) = \sum\limits_{j=1}^{n}(|A'_j| - |A'_j \cap \bigcap\limits_{i=1}^{n}(A'_i + H_a)|)$. Furthermore, if H_a is nontrivial, then $\phi_a(x) \in \phi_a(A'_i)$ for every $i \in \{1, \ldots, n\}$ and every $x \in S \setminus S''$.

Note that Theorem 2.1 implies $|\sum\limits_{i=1}^{n} A'_i| \geq \min\{|G|, |S'| - n + 1\}$ unless $N(A', H_a) > 0$ and H_a is a proper, nontrivial subgroup. Let $\rho = Nn|H_a| - |S'| + e$, where $N = N(A', H_a)$ and $e = E(A', H_a)$, be the number of H_a-holes contained among the sets $A'_j \cap \bigcap\limits_{i=1}^{n}(A'_i + H_a)$, $j = 1, \ldots, n$. Also observe that if Theorem 2.1 does not hold with H_a trivial, then $(e + (N-1)n + 1)|H_a| \leq |S'| - n$ follows, implying $Nn|H_a| - |S'| \leq n(|H_a| - 1) - |H_a| - e|H_a|$, which from the previous sentences implies

$$\rho < (n - 1 - e)(|H_a| - 1) \leq (n-1)(|H_a| - 1),$$

mirroring the bound obtained from Kneser's Theorem discussed earlier.

We will need the following draining theorem for n-set partitions [19].

Theorem 2.2. *Let S be a finite sequence of elements from an abelian group G. If S has an n-set partition, $A = A_1, \ldots, A_n$, such that*

$$|\sum_{i=1}^{n} A_i| \geq \sum_{i=1}^{n} |A_i| - n + 1,$$

then there exists a subsequence S' of S, with length $|S'| \leq \max\{|S| - n + 1,\, 2n\}$, and with an n-set partition, $A' = A'_1, \ldots, A'_n$, such that $|\sum\limits_{i=1}^{n} A'_i| \geq \sum\limits_{i=1}^{n} |A_i| - n + 1$. Furthermore, if $||A_i| - |A_j|| \leq 1$ for all i and j, or if $|A_i| \geq 3$ for all i, then $A'_i \subseteq A_i$.

Finally, we conclude with the following well-known and basic theorem bounding the real roots of a polynomial with real coefficients.

Theorem 2.3. *Let $P(x)$ be a polynomial with real coefficients and positive leading coefficient, and let a be a real number. If $a > 0$, and all nonzero terms of $P(x)/(x-a)$, including remainder (computed by polynomial division), are positive, then a is an upper bound for all real roots of $P(x)$.*

Proof. Let $P(x) = Q(x)(x-a) + r$, with $r \in \mathbb{R}$. Since all nonzero terms of $P(x)/(x-a)$, including remainder (computed by polynomial division), are positive, it follows that $r \geq 0$ and $Q(x) > 0$ for all real $x > 0$. Thus, since for $x > a > 0$ we have $x-a > 0$, it follows that $P(x) = Q(x)(x-a)+r > 0$ for $x > a$. $\square$

3 The Proof

In view of the results of Kisin [31] mentioned in the introduction, it follows that Conjecture 1.1 is known for $m < 30$, as well as for $m = 2^5 = 32$, $m = 5 \cdot 7 = 35$, and $m = 2 \cdot 19 = 38$. We begin by proving several lemmas relating the sizes of two different binomial coefficients. Note that in view of the first sentence of this section, then Lemma 3.2 and Theorem 1.1 together imply Conjecture 1.1 for $n \leq 6\frac{1}{3}m$. Both Lemmas 3.1 and 3.2 are straightforward computations, best done with machine assistance, but for the benefit of the reader we include many of the details.

Lemma 3.1. *If $m \geq 30$ and n are integers with $2m-1 \leq n \leq 3m+\lceil\frac{2m-1}{3}\rceil-2$, then $\binom{n-m}{\lceil\frac{m}{2}\rceil} > 2\binom{\lceil\frac{n}{2}\rceil}{m}$.*

Proof. Let $R(n,m) = \binom{n-m}{\lceil\frac{m}{2}\rceil}/2\binom{\frac{n+1}{2}}{m} = \frac{(n-m)...(n-m-\lceil\frac{m}{2}\rceil+1)(m)...(\lceil\frac{m}{2}\rceil+1)}{2(\frac{n+1}{2})...(\frac{n+1}{2}-m+1)}$. Since $\binom{\frac{n+1}{2}}{m} \geq \binom{\lceil\frac{n}{2}\rceil}{m}$, then it suffices to show $R(n,m) > 1$. We begin by showing that $R(n,m) \geq R(n+2,m)$.

Let $Q(n,m) = \frac{(n-m-\frac{m+1}{2}+2)(n-m-\frac{m+1}{2}+1)(\frac{n+1}{2}+1)}{(n-m+2)(n-m+1)(\frac{n+1}{2}-m+1)} \leq R(n,m)/R(n+2,m)$. To show $R(n,m) \geq R(n+2,m)$, we will show that $Q(n,m) \geq 1$, i.e. (by multiplying out the denominator, expanding and collecting terms) that

$$4(m-1)n^2 - (11m^2 - 12m + 17)n + (8m^3 - 9m^2 + 16m - 15) \geq 0.$$

This will occur if both roots of the above polynomial are imaginary, which by the quadratic formula occurs when

$$m^4 - \frac{8}{7}m^3 - \frac{118}{7}m^2 - \frac{88}{7}m - 7 > 0. \tag{1}$$

However, in view of Theorem 2.3 it follows that the roots of the polynomial $m^4 - \frac{8}{7}m^3 - \frac{118}{7}m^2 - \frac{88}{7}m - 7$ are bounded from above by 6. Consequently (1) holds for $m \geq 7$, and we can assume $R(n,m) \geq R(n+2,m)$.

Since $R(n,m) \geq R(n+2,m)$, it suffices to show $R(3\frac{2}{3}m+b, m) > 1$ for $b = -2 + (\lceil \frac{2m-1}{3} \rceil - \frac{2}{3}m)$ and $b = -3 + (\lceil \frac{2m-1}{3} \rceil - \frac{2}{3}m)$. Note $b \in \{-\frac{5}{3}, -\frac{6}{3}, -\frac{7}{3}, -\frac{8}{3}, -\frac{9}{3}, -\frac{10}{3}\}$. Let $S(m) = R(3\frac{2}{3}m+b, m)$. Next we show that $S(m+6) \geq S(m)$. Note that computing $S(m)$ for each $m \in \{30, \ldots, 35\}$ and both possible values for b shows that $S(m) > 1$ for $m \in \{30, \ldots, 35\}$. Hence the proof will be complete once we have shown that $S(m+6) \geq S(m)$.

Let $P(m) = \frac{(\frac{8}{3}m+b+16)...(\frac{8}{3}m+b+1)(m+6)...(m+1)(\frac{5}{6}m+\frac{b+1}{2}+5)...(\frac{5}{6}m+\frac{b+1}{2}+1)}{(\frac{11}{6}m+\frac{b+1}{2}+11)...(\frac{11}{6}m+\frac{b+1}{2}+1)(\frac{m+1}{2}+3)...(\frac{m+1}{2}+1)(\frac{13}{6}m+b+13)...(\frac{13}{6}m+b+1)} \leq S(m+6)/S(m)$. To see that $S(m+6) \geq S(m)$, we will show that $P(m) \geq 1$. By multiplying out denominators, bringing all terms to the left hand side, expanding and collecting terms, and rounding coefficients down, it follows that it suffices to show $-3 \cdot 10^{17} - 4 \cdot 10^{18}m - 3 \cdot 10^{19}m^2 - 2 \cdot 10^{20}m^3 - 4 \cdot 10^{20}m^4 - 7 \cdot 10^{20}m^5 - 2 \cdot 10^{21}m^6 - 2 \cdot 10^{21}m^7 - 2 \cdot 10^{21}m^8 - 2 \cdot 10^{21}m^9 - 8 \cdot 10^{20}m^{10} - 5 \cdot 10^{20}m^{11} - 2 \cdot 10^{20}m^{12} - 8 \cdot 10^{19}m^{13} - 3 \cdot 10^{19}m^{14} - 7 \cdot 10^{18}m^{15} - 2 \cdot 10^{18}m^{16} - 4 \cdot 10^{17}m^{17} - 6 \cdot 10^{16}m^{18} - 7 \cdot 10^{15}m^{19} - 7 \cdot 10^{14}m^{20} - 5 \cdot 10^{13}m^{21} - 2 \cdot 10^{12}m^{22} - 4 \cdot 10^{10}m^{23} + 7 \cdot 10^{8}m^{24} + 2 \cdot 10^{8}m^{25} + 10^{7}m^{26} + 3 \cdot 10^{5}m^{27} > 0$, in order to show $P(m) \geq 1$ (the rounded polynomial just given is strictly less, for positive m, than the corresponding polynomial for each value of $b \in \{-\frac{5}{3}, -\frac{6}{3}, -\frac{7}{3}, -\frac{8}{3}, -\frac{9}{3}, -\frac{10}{3}\}$ obtained by algebraic manipulation). However, in view of Theorem 2.3, it follows that the roots of the polynomial from the previous sentence are all bounded from above by 23, implying that the inequality from the last sentence holds for $m \geq 24$, which completes the proof. $\square$

Lemma 3.2. *If $m \geq 30$ and n are integers either with $2m-1 \leq n \leq 6\frac{1}{3}m$, $m \neq 32$, $m \neq 35$, and $m \neq 38$, or else with $2m-1 \leq n \leq 6\frac{1}{3}m - 6$, then $\binom{n-m}{\lceil \frac{2m-1}{3} \rceil} > 2\binom{\lceil \frac{n}{2} \rceil}{m}$.*

Proof. Let $R(n,m) = \binom{n-m}{\lceil \frac{2m-1}{3} \rceil} / 2\binom{\frac{n+1}{2}}{m} = \frac{(n-m)\ldots(n-m-\lceil \frac{2m-1}{3} \rceil+1)(m)\ldots(\lceil \frac{2m-1}{3} \rceil+1)}{2(\frac{n+1}{2})\ldots(\frac{n+1}{2}-m+1)}$. Since $\binom{\frac{n+1}{2}}{m} \geq \binom{\lceil \frac{n}{2} \rceil}{m}$, then it suffices to show $R(n,m) > 1$. We begin by showing that $R(n,m) \geq R(n+2,m)$.

Let $Q(n,m) = \frac{(n-m-\frac{2m+1}{3}+2)(n-m-\frac{2m+1}{3}+1)(\frac{n+1}{2}+1)}{(n-m+2)(n-m+1)(\frac{n+1}{2}-m+1)} \leq R(n,m)/R(n+2,m)$. To show $R(n,m) \geq R(n+2,m)$, we will show that $Q(n,m) \geq 1$, i.e. (by multiplying out the denominator, expanding and collecting terms) that

$$3(m-1)n^2 - (10m^2 - 5m + 13)n + (9m^3 - 3m^2 + 6m - 12) \geq 0.$$

This will occur if both roots of the above polynomial are imaginary, which by the quadratic formula occurs when

$$m^4 - \frac{11}{2}m^3 - \frac{177}{8}m^2 - \frac{43}{4}m - \frac{25}{8} > 0. \tag{2}$$

However, in view of Theorem 2.3 it follows that the roots of the polynomial $m^4 - \frac{11}{2}m^3 - \frac{177}{8}m^2 - \frac{43}{4}m - \frac{25}{8}$ are bounded from above by 9. Consequently (2) holds for $m \geq 10$, and we can assume $R(n,m) \geq R(n+2,m)$.

First assume that $n \leq 6\frac{1}{3}m$ with $m \neq 32$, $m \neq 35$, and $m \neq 38$. Since $R(n,m) \geq R(n+2,m)$, it suffices to show $R(6\frac{1}{3}m + b, m) > 1$ for $b = (\lfloor 6\frac{1}{3}m \rfloor - 6\frac{1}{3}m)$ and $b = -1 + (\lfloor 6\frac{1}{3}m \rfloor - 6\frac{1}{3}m)$. Note $b \in \{0, -\frac{1}{3}, -\frac{2}{3}, -\frac{3}{3}, -\frac{4}{3}, -\frac{5}{3}\}$. Let $S(m) = R(6\frac{1}{3}m + b, m)$. Next we show that $S(m+6) \geq S(m)$ for $m \geq 43$. Note that computing $S(m)$ for each $m \leq 48$, $m \neq 32$, $m \neq 35$, $m \neq 38$, and both possible values for b, shows that $S(m) > 1$ for $m \leq 48$, $m \neq 32$, $m \neq 35$, $m \neq 38$. Hence the first part of the lemma will be complete once we have shown that $S(m+6) \geq S(m)$ for $m \geq 43$.

Let $P(m) = \frac{(\frac{16}{3}m+b+32)...(\frac{16}{3}m+b+1)(m+6)...(m+1)(\frac{13}{6}m+\frac{b+1}{2}+13)...(\frac{13}{6}m+\frac{b+1}{2}+1)}{(\frac{19}{6}m+\frac{b+1}{2}+19)...(\frac{19}{6}m+\frac{b+1}{2}+1)(\frac{2m+1}{3}+4)...(\frac{2m+1}{3}+1)(\frac{14}{3}m+b+\frac{1}{3}+28)...(\frac{14}{3}m+b+\frac{1}{3}+1)}$. Note that $P(m) \leq S(m+6)/S(m)$ for $m \geq 43$. To see that $S(m+6) \geq S(m)$, it suffices to show $P(m) \geq 1$. The proof proceeds as in the previous lemma. The case with $n \leq 6\frac{1}{3}m - 6$ is handled similarly. $\square$

Lemma 3.3. *Let n, m, and x be positive integers. If $n \geq \frac{3}{2}m - 1$, then $3^x\binom{n}{m} \geq \binom{n+x}{m}$.*

Proof. Observe that the following binomial identity holds:

$$\binom{n}{m} = \frac{n-m+1}{m}\binom{n}{m-1}. \tag{3}$$

Since $n \geq \frac{3}{2}m - 1$, then (3) implies that $2\binom{n+x'}{m} \geq \binom{n+x'}{m-1}$, for $x' \geq 0$. Hence from the Pascal identity, it follows that

$$3\binom{n+x'}{m} \geq \binom{n+x'}{m} + \binom{n+x'}{m-1} = \binom{n+x'+1}{m},$$

for $x' \geq 0$. Iterating the above inequality for $x' = 0, \ldots, x-1$ yields $3^x\binom{n}{m} \geq \binom{n+x}{m}$. $\square$

We now proceed with the proof of Theorem 1.1.

Proof. The proof will be divided into several steps. For our main method to work, we will need the existence of a sufficiently compressed $\lceil\frac{n}{2}\rceil$-set partition. Thus we will first handle several special and highly restrictive sequences S which do not admit such a compressed set partition.

Let $\mathscr{Z}_m(S)$ denote the number of m-term zero-sum subsequences of S. Note that from the Erdős-Ginzburg-Ziv Theorem it follows trivially that $\mathscr{Z}_m(S) \geq n - 2m + 2$. Thus $\mathscr{Z}_m(S) \geq \binom{\lceil\frac{n}{2}\rceil}{m} + \binom{\lfloor\frac{n}{2}\rfloor}{m}$ holds for $n \leq 2m$. Consequently, inductively assume $\mathscr{Z}_m(S') \geq \min\{\binom{\lceil\frac{n}{2}\rceil}{m} + \binom{\lfloor\frac{n}{2}\rfloor}{m}, \binom{n-m}{\lceil\frac{2m-1}{3}\rceil}\}$ holds for any sequence S' of n' terms from an abelian group of order m provided $n' < n$, and also assume that $n \geq 2m + 1$. In view of the results of Kisin [31], we may assume that m is composite.

Step 1 (S essentially monochromatic): Suppose that there is a term x of S with multiplicity at least $\lceil \frac{n}{2} \rceil$. Then there will be at least $\binom{\lceil \frac{n}{2} \rceil - 1}{m-1}$ m-term monochromatic (and hence also zero-sum) subsequences of S that include the term x. By induction hypothesis there are at least $\min\{\binom{\lceil \frac{n-1}{2} \rceil}{m} + \binom{\lfloor \frac{n-1}{2} \rfloor}{m}, \binom{n-m-1}{\lceil \frac{2m-1}{3} \rceil}\}$ m-term zero-sum subsequences that do not include the term x. Hence there are in total at least $\min\{\binom{\lceil \frac{n}{2} \rceil - 1}{m-1} + \binom{\lceil \frac{n-1}{2} \rceil}{m} + \binom{\lfloor \frac{n-1}{2} \rfloor}{m}, \binom{\lceil \frac{n}{2} \rceil - 1}{m-1} + \binom{n-m-1}{\lceil \frac{2m-1}{3} \rceil}\}$ m-term zero-sum subsequences. By the Pascal identity for binomial coefficients, it follows that

$$\binom{\lceil \frac{n}{2} \rceil - 1}{m-1} + \binom{\lceil \frac{n-1}{2} \rceil}{m} + \binom{\lfloor \frac{n-1}{2} \rfloor}{m} = \binom{\lceil \frac{n}{2} \rceil - 1}{m-1} + \binom{\lceil \frac{n}{2} \rceil - 1}{m} + \binom{\lfloor \frac{n}{2} \rfloor}{m} = \binom{\lceil \frac{n}{2} \rceil}{m} + \binom{\lfloor \frac{n}{2} \rfloor}{m}.$$

Thus the proof is complete unless

$$\binom{n-m-1}{\lceil \frac{2m-1}{3} \rceil} < \binom{\lceil \frac{n-1}{2} \rceil}{m} + \binom{\lfloor \frac{n-1}{2} \rfloor}{m}, \tag{4}$$

and

$$\binom{\lceil \frac{n}{2} \rceil - 1}{m-1} + \binom{n-m-1}{\lceil \frac{2m-1}{3} \rceil} < \binom{n-m}{\lceil \frac{2m-1}{3} \rceil}.$$

From the above inequality and the Pascal identity, it follows that

$$\binom{\lceil \frac{n}{2} \rceil - 1}{m-1} < \binom{n-m-1}{\lceil \frac{2m-1}{3} \rceil - 1}. \tag{5}$$

From (4) and Lemma 3.2, it follows that $n-1 > 6\frac{1}{3}m - 6$. Applying the binomial identity given in (3) to (5), it follows that

$$\binom{\lceil \frac{n}{2} \rceil - 1}{m-1} < \frac{\lceil \frac{2m-1}{3} \rceil}{(n-m-\lceil \frac{2m-1}{3} \rceil)} \binom{n-m-1}{\lceil \frac{2m-1}{3} \rceil}, \tag{6}$$

and that

$$\binom{\lceil \frac{n}{2} \rceil - 1}{m} < \frac{(\lceil \frac{n}{2} \rceil - m)}{m} \cdot \frac{\lceil \frac{2m-1}{3} \rceil}{(n-m-\lceil \frac{2m-1}{3} \rceil)} \binom{n-m-1}{\lceil \frac{2m-1}{3} \rceil}. \tag{7}$$

If n is odd, then (4) implies $\binom{n-m-1}{\lceil \frac{2m-1}{3} \rceil} < \binom{\lceil \frac{n}{2} \rceil - 1}{m} + \binom{\lceil \frac{n}{2} \rceil - 1}{m}$, and if n is even, then (4) and the Pascal identity imply $\binom{n-m-1}{\lceil \frac{2m-1}{3} \rceil} < \binom{\lceil \frac{n}{2} \rceil}{m} + \binom{\lceil \frac{n}{2} \rceil - 1}{m} = \binom{\lceil \frac{n}{2} \rceil - 1}{m} + \binom{\lceil \frac{n}{2} \rceil - 1}{m-1} + \binom{\lceil \frac{n}{2} \rceil - 1}{m}$. Hence from (6) and (7), it follows that

$$\binom{n-m-1}{\lceil\frac{2m-1}{3}\rceil} < \left(2\cdot\frac{(\lceil\frac{n}{2}\rceil-m)}{m}\cdot\frac{\lceil\frac{2m-1}{3}\rceil}{(n-m-\lceil\frac{2m-1}{3}\rceil)}+\frac{\lceil\frac{2m-1}{3}\rceil}{(n-m-\lceil\frac{2m-1}{3}\rceil)}\right)\binom{n-m-1}{\lceil\frac{2m-1}{3}\rceil},$$

which in turn implies that

$$1<\frac{2(\frac{n+1}{2}-m)\cdot\frac{2m+1}{3}+m\cdot\frac{2m+1}{3}}{m\cdot(n-m-\frac{2m+1}{3})}.$$

From the above inequality, it follows that $(m-1)n < 3m^2+2m+1$, implying $n < 3m+5+\frac{6}{m-1}$, which contradicts that $n-1 > 6\frac{1}{3}m-6$ and $m \geq 30$. So we may assume that the multiplicity of every term x of S is at most $\lceil\frac{n}{2}\rceil-1$.

Step 2 (S essentially dichromatic): Suppose that every term of S, with at most $\max\{m-\frac{m}{p},\, \lfloor\frac{2m-4}{3}\rfloor\}$ exceptions if $n \geq 3m+\lceil\frac{2m-1}{3}\rceil-1$, and with at most $m-\frac{m}{p}$ exceptions if $n \leq 3m+\lceil\frac{2m-1}{3}\rceil-2$, is equal to one of two elements $x,\, y \in G$, where p is the smallest prime divisor of m. Let n_x and n_y denote the respective multiplicities of x and y in S. Rearrange the terms of S so that all the terms equal to x proceed all the terms equal to y, which in turn proceed all terms equal to neither x nor y, and let $x_1,\ldots,x_n$ be the resulting sequence. For $i \in \{1,\ldots,\lfloor\frac{n}{2}\rfloor\}$, let $A_i=\{x_i,\, x_{i+\lceil\frac{n}{2}\rceil}\}$, and if n is odd, then let $A_{\lceil\frac{n}{2}\rceil}=\{x_{\lceil\frac{n}{2}\rceil}\}$. Then in view of Step 1, it follows that $A=A_1,\ldots,A_{\lceil\frac{n}{2}\rceil}$ is an $\lceil\frac{n}{2}\rceil$-set partition of S such that either $x\in A_i$ or $y \in A_i$ holds for every set A_i.

There are $\binom{\lfloor\frac{n}{2}\rfloor}{m}$ ways to choose m sets A_i from A all with $|A_i|=2$, and (in case n odd) there are $\binom{\lceil\frac{n}{2}\rceil-1}{m-1}$ ways to choose m sets A_i from A that include the set $A_{\lceil\frac{n}{2}\rceil}$ of cardinality one. Consequently, if we can show that any such selection $A_{i_1},\ldots,A_{i_m}$ has a set A_{i_k} such that $0\in z+\sum\limits_{\substack{j=1\\ j\neq k}}^{m}A_{i_j}$ for every $z\in A_{i_k}$ (in which case we will say that the selection $A_{i_1},\ldots,A_{i_m}$ is *good*), then there will be (in case n even) at least $2\binom{\lfloor\frac{n}{2}\rfloor}{m}=\binom{\lfloor\frac{n}{2}\rfloor}{m}+\binom{\lceil\frac{n}{2}\rceil}{m}$ m-term zero-sum subsequences, and (in case n odd), in view of the Pascal identity, at least $2\binom{\lfloor\frac{n}{2}\rfloor}{m}+\binom{\lceil\frac{n}{2}\rceil-1}{m-1}=\binom{\lfloor\frac{n}{2}\rfloor}{m}+\binom{\lceil\frac{n}{2}\rceil-1}{m}+\binom{\lceil\frac{n}{2}\rceil-1}{m-1}=\binom{\lfloor\frac{n}{2}\rfloor}{m}+\binom{\lceil\frac{n}{2}\rceil}{m}$

m-term zero-sum subsequences, whence the proof is complete. We proceed to show this is the case, except for a highly restrictive sequence that we handle separately afterwards.

If the selection $A_{i_1}, \ldots, A_{i_m}$ contains the set $A_{\lceil \frac{n}{2} \rceil}$ and n is odd, then let $A_{i_k} = A_{\lceil \frac{n}{2} \rceil}$, and otherwise let A_{i_k} be a set $A_{i_j} = \{x, y\}$ (such a set exists, since at most $\max\{m - \frac{m}{p}, \lfloor \frac{2m-4}{3} \rfloor\} < m$ terms of S are equal to neither x nor y). If $|\sum_{\substack{j=1 \\ j \neq k}}^{m} A_{i_j}| \geq \sum_{\substack{j=1 \\ j \neq k}}^{m} |A_j| - (m-1) + 1 = m$, then for each $z \in A_{i_k}$ we can select a term from each of the A_{i_j}, $j \neq k$, so that the sum of the $m-1$ selected terms from the A_{i_j}, $j \neq k$, is the additive inverse of z, whence we see that the selection $A_{i_1}, \ldots, A_{i_m}$ is good. Otherwise, from Kneser's Theorem it follows that $\sum_{\substack{j=1 \\ j \neq k}}^{m} A_{i_j}$ is maximally H_a-periodic, with H_a of index a and $1 < a < m$.

Suppose that $\phi_a(x) = \phi_a(y)$, i.e. that x and y are from the same H_a-coset. Hence, since every set A_{i_j} contains either x or y, it follows that every set A_{i_j} contains a representative from the coset $x + H_a$. Hence, since $\sum_{\substack{j=1 \\ j \neq k}}^{m} A_{i_j}$ is H_a-periodic, it follows that $0 \in H_a = mx + H_a \subset z + \sum_{\substack{j=1 \\ j \neq k}}^{m} A_{i_j}$ for $z \in A_{i_k} \subseteq \{x, y\}$, and the proof is again complete. So we may assume that $\phi_a(x) \neq \phi_a(y)$.

If there are at most $m - \frac{m}{p}$ terms of S equal to neither x nor y, then there must be at least $a-1$ sets A_{i_j}, $j \neq k$, with $A_{i_j} = \{x, y\}$, and hence, since $\phi_a(x) \neq \phi_a(y)$, at least $a-1$ sets A_{i_j} with $|\phi_a(A_{i_j})| = 2$. On the other hand, if there are at most $\lfloor \frac{2m-4}{3} \rfloor$ terms of S equal to neither x nor y, then either there likewise must be at least $a-1$ sets A_{i_j} with $|\phi_a(A_{i_j})| = 2$, or else $|H_a| = 2$, and there are at least $\frac{m}{2} + 2$ sets A_{i_j} with $A_{i_j} \neq \{x, y\}$ and A_{i_j} contained in an H_a-coset. If the former holds, then from Kneser's Theorem it follows that $|\sum_{\substack{j=1 \\ j \neq k}}^{m} A_j| \geq |H_a|(\sum_{\substack{j=1 \\ j \neq k}}^{m} |\phi_a(A_j)| - (m-1) + 1) \geq m$, and the proof is again complete. Therefore we may instead assume the latter. Consequently we can assume $n \geq 3m + \lceil \frac{2m-1}{3} \rceil - 1$, that m is even, and that there are at least $m - \frac{m}{p} + 1 = \frac{m}{2} + 1$ terms t of S with $t \notin \{x, y\}$.

Suppose that $x-y$ generates a proper subgroup H_b of index b. Since there are at most $\lfloor\frac{2m-4}{3}\rfloor$ terms of S equal to neither x nor y, and since there are at least $\frac{m}{2}+1$ sets A_i with $A_i \neq \{x,y\}$ and A_i an H_a-coset, then we can re-index the sets A_i so that $A_i=\{x,y\}$ for $i \leq \lfloor\frac{n}{2}\rfloor - \lfloor\frac{2m-4}{3}\rfloor$, and so that A_i is an H_a-coset for $\lfloor\frac{n}{2}\rfloor - \lfloor\frac{2m-4}{3}\rfloor + 1 \leq i \leq \lfloor\frac{n}{2}\rfloor - \lfloor\frac{2m-4}{3}\rfloor + \frac{m}{2} + 1$. Let $A_{i'_1}, \ldots, A_{i'_m}$ be a selection of m sets A_i all with $i \leq \lfloor\frac{n}{2}\rfloor - \lfloor\frac{2m-4}{3}\rfloor + \frac{m}{2} + 1 = \lfloor\frac{n}{2}\rfloor - \lfloor\frac{m-8}{6}\rfloor + 1$.

If $A_{i'_j} = \{x,y\}$ for all j, then $\sum\limits_{j=1}^{\frac{m}{b}-1} A_{i'_j}$ is an H_b-coset, whence there will be at least $2^{m-\frac{m}{b}+1} \geq 2^{\frac{m}{2}}$ ways to a select a term from each set $A_{i'_j}$ and get an m-term zero-sum subsequence. Next suppose that at least one of the $A_{i'_j}$, say w.l.o.g. $A_{i'_1}$, is an H_a-coset. Since at most $\frac{m}{2}+1$ of the sets $A_{i'_j}$ can be H_a-cosets, it follows that there are at least $\frac{m}{2}-1$ indices j with $A_{i'_j} = \{x,y\}$. Re-index so that $A_{i'_j} = \{x,y\}$ for $2 \leq i \leq \frac{m}{2}$. Hence $\sum\limits_{j=1}^{\frac{m}{2}} A_{i'_j}$ is an (H_a+H_b)-coset. Thus, since every set $A_{i'_j}$ contains either x or y, then it follows that every set $A_{i'_j}$ is contained in the same (H_a+H_b)-coset $x+H_a+H_b$, whence it follows that there will also be at least $2^{\frac{m}{2}}$ ways to a select a term from each set $A_{i'_j}$ and get an m-term zero-sum subsequence. Thus we conclude that there are at least

$$2^{\frac{m}{2}}\binom{\lfloor\frac{n}{2}\rfloor - \lfloor\frac{m-8}{6}\rfloor + 1}{m} \tag{8}$$

m-term zero-sum subsequences. Since $n \geq 3m + \lceil\frac{2m-1}{3}\rceil - 1$, then it follows in view of Lemma 3.3 that $3^x\binom{\lfloor\frac{n}{2}\rfloor - \lfloor\frac{m-8}{6}\rfloor+1}{m} \geq \binom{\lfloor\frac{n}{2}\rfloor - \lfloor\frac{m-8}{6}\rfloor+1+x}{m}$. Hence from (8) it follows that there are at least

$$2^{\frac{m}{2}}\binom{\lfloor\frac{n}{2}\rfloor - \lfloor\frac{m-8}{6}\rfloor + 1}{m} \geq 2\cdot 4^{\lfloor\frac{m-2}{4}\rfloor}\binom{\lfloor\frac{n}{2}\rfloor - \lfloor\frac{m-8}{6}\rfloor + 1}{m} \geq 2\cdot 3^{\lfloor\frac{m-2}{4}\rfloor}\binom{\lfloor\frac{n}{2}\rfloor - \lfloor\frac{m-8}{6}\rfloor + 1}{m} \geq$$
$$2\binom{\lfloor\frac{n}{2}\rfloor - \lfloor\frac{m-8}{6}\rfloor + \lfloor\frac{m-2}{4}\rfloor + 1}{m} \geq 2\binom{\lceil\frac{n}{2}\rceil}{m} \geq \binom{\lceil\frac{n}{2}\rceil}{m} + \binom{\lfloor\frac{n}{2}\rfloor}{m}$$

m-term zero-sum subsequences, whence the proof is complete. So we may assume that $x-y$ generates G, implying G is cyclic of order m.

Suppose $n_x \leq \lfloor\frac{n}{2}\rfloor - \frac{m}{2}$. Re-index the terms x_i in the sequence $x_1, \ldots, x_n$ with $x_i \notin \{x,y\}$ (leaving unchanged the terms $x_i \in \{x,y\}$) so that all terms x_i with $x_i \notin \{x, y, y+\frac{m}{2}\}$ occur in a consecutive block at the very end of the sequence. Then, since in a cyclic group there is a unique subgroup of order two, it follows that either every set A_i will contain a representative from the common H_a-coset $y + H_a$, or else every set A_i contained in an $H_{a'}$-coset with $|H_{a'}| = 2$ and $i \leq \lfloor \frac{n}{2} \rfloor$ must contain x. In the latter case, since $n_x \leq \lfloor \frac{n}{2} \rfloor - \frac{m}{2}$, it follows that there are at most $\lfloor \frac{2m-4}{3} \rfloor - \frac{m}{2} + 1 < \frac{m}{2} + 2$ sets A_i contained in an $H_{a'}$-coset with $|H_{a'}| = 2$, which reduces to a case handled in the fifth paragraph of Step 2. Therefore we may assume the former case holds. From previous work, we know that any selection $A_{i_1}, \ldots, A_{i_m}$ is good unless $\sum\limits_{\substack{j=1 \\ j \neq k}}^{m} A_{i_j}$ is maximally $H_{a'}$-periodic with $|H_{a'}| = 2$ and $|\sum\limits_{\substack{j=1 \\ j \neq k}}^{m} A_{i_j}| < \sum\limits_{\substack{j=1 \\ j \neq k}}^{m} |A_{i_j}| - (m-1) + 1$. However, since there is a unique subgroup H_a of order two, it follows that $H_{a'} = H_a$. Hence, since every set A_i contains a representative from the common H_a-coset $y + H_a$, and since $\sum\limits_{j=1}^{m} A_{i_j}$ is H_a-periodic, it follows that $0 \in \sum\limits_{j=1}^{m} A_{i_j}$. Since $|\sum\limits_{\substack{j=1 \\ j \neq k}}^{m} A_{i_j}| < \sum\limits_{\substack{j=1 \\ j \neq k}}^{m} |A_{i_j}| - (m-1) + 1$, and since $|A_{i_j}| = 2$ for $j \neq i_k$, it follows from Proposition 2.2 that there exists A_{i_l} with $l \neq k$ such that $|\sum\limits_{\substack{j=1 \\ j \neq l}}^{m} A_{i_j}| = |\sum\limits_{j=1}^{m} A_{i_j}|$, whence it follows that every $z \in \sum\limits_{j=1}^{m} A_{i_j}$ can be represented in at least two different ways, including $0 \in \sum\limits_{j=1}^{m} A_{i_j}$. Thus every selection $A_{i_1}, \ldots, A_{i_m}$ is good, completing the proof. So we may assume that $n_x \geq \lfloor \frac{n}{2} \rfloor - \frac{m}{2} + 1$.

Re-index the terms x_i in the sequence $x_1, \ldots, x_n$ with $x_i \notin \{x, y\}$ (leaving unchanged the terms $x_i \in \{x, y\}$) so that all terms x_i with $x_i = x + \frac{m}{2}$ occur in a consecutive block at the very end of the sequence. Since $n_x \geq \lfloor \frac{n}{2} \rfloor - \frac{m}{2} + 1$, and since there are at least $m - \frac{m}{p} + 1 = \frac{m}{2} + 1$ terms t with $t \notin \{x, y\}$, it follows that $A_{n_x} = \{x, t\}$ with $t \notin \{x, y\}$. If n is odd, then modify the definition of the set partition $A_1, \ldots, A_{\lceil \frac{n}{2} \rceil}$ by swapping the term equal to x in A_{n_x} with the term equal to y in $A_{\lceil \frac{n}{2} \rceil}$. The proof now proceeds as in the above paragraph with the roles of x and y interchanged, completing Step 2. So we may assume that given any two elements x, $y \in G$, then there are at least $m - \frac{m}{p} + 1$ terms of S equal to neither x nor y, and, if $n \geq 3m + \lceil \frac{2m-1}{3} \rceil - 1$, then there are at least $\lfloor \frac{2m-1}{3} \rfloor$ terms of S equal to neither x nor y.

Step 3 ($|S| \leq 3m + \lceil \frac{2m-1}{3} \rceil - 2$): Suppose that $n \leq 3m + \lceil \frac{2m-1}{3} \rceil - 2$. In view of Steps 1 and 2 and Proposition 2.1 applied with $n_1 = n - m + 1$ and $n_2 = \lfloor \frac{m}{2} \rfloor$, it follows that there exists an $(n-m+1)$-set partition $P = P_1, \ldots, P_{n-m+1}$ of S with $|P_i| = 1$ for $i > \lfloor \frac{m}{2} \rfloor$. Let $P' = P_1, \ldots, P_{\lfloor \frac{m}{2} \rfloor}$, and let S' be the subsequence partitioned by the $\lfloor \frac{m}{2} \rfloor$-set partition P'. Apply Theorem 2.1 to the subsequence S' of S with $\lfloor \frac{m}{2} \rfloor$-set partition P', and let $A = A_1, \ldots, A_{\lfloor \frac{m}{2} \rfloor}$ be the resulting set partition, and H_a the corresponding subgroup of index a.

Suppose that $|\sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} A_i| \geq m = \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} |A_i| - \lfloor \frac{m}{2} \rfloor + 1$. Then applying Theorem 2.2 to A and S' yields a subsequence S'' of S' of length m with an $\lfloor \frac{m}{2} \rfloor$-set partition $A' = A'_1, \ldots, A'_{\lfloor \frac{m}{2} \rfloor}$ satisfying $|\sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} A'_i| \geq m$. Then given any $\lceil \frac{m}{2} \rceil$-term subsequence T of $S \setminus S''$, we can find a selection of $\lfloor \frac{m}{2} \rfloor$ terms from the $A'_1, \ldots, A'_{\lfloor \frac{m}{2} \rfloor}$ that sum to the additive inverse of the sum of the terms from T. Consequently, there will be least $\binom{n-m}{\lceil \frac{m}{2} \rceil}$ m-term zero-sum subsequences. Thus, since $n \leq 3m + \lceil \frac{2m-1}{3} \rceil - 2$, it follows in view of Lemma 3.1 that the proof is complete. So we may assume that $|\sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} A_i| < m = \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} |A_i| - \lfloor \frac{m}{2} \rfloor + 1$.

Thus from Theorem 2.1 it follows that $N(A', H_a) = 1$ and $E(A', H_a) \leq a - 2$, with H_a a nontrivial, proper subgroup. Hence all but at most $a - 2$ terms of S are from the same H_a-coset, say $\alpha + H_a$. Hence, let H_b be a minimal cardinality nontrivial, proper subgroup of index b such that all but at most $b - 2$ terms of S are all from the same H_b-coset, say $\beta + H_b$, and such that there exists an $(n-m+1)$-set partition $B = B_1, \ldots, B_{n-m+1}$ of the terms of S from $\beta + H_a$ with $|B_i| = 1$ for $i > \lfloor \frac{m}{2} \rfloor$ (in view of the previous two sentences, and taking $B_i = A'_i \cap (\alpha + H_a)$ for $i \leq \lfloor \frac{m}{2} \rfloor$, and appending on an additional $n - m + 1 - \lfloor \frac{m}{2} \rfloor$ singleton sets using the terms from $S \setminus S''$, it follows that such a subgroup exists). We may w.l.o.g. by translation assume $\beta = 0$. Let S_b be the subsequence of S consisting of terms from H_b, and let S'_b be the subsequence of S_b partitioned by the set partition $B' = B_1, \ldots, B_{\lfloor \frac{m}{2} \rfloor}$. Apply Theorem 2.1 to the subsequence S'_b of S_b with $\lfloor \frac{m}{2} \rfloor$-set partition B' and with $G = H_b$, and let $B'' = B'_1, \ldots, B'_{\lfloor \frac{m}{2} \rfloor}$ be the resulting set partition and H_{kb} the corresponding subgroup with $[H_b : H_{kb}] = k$. If $N(B'', H_{kb}) = 1$ and $E(B'', H_{kb}) \leq k - 2$, with H_{kb} a nontrivial, proper subgroup, then all but at most $k - 2 + b - 2 \leq kb - 2$ terms of S will all be from the same H_{kb}-coset, contradicting the minimality of H_b (the needed $(n - m + 1)$-set partition can be induced from the set partition B'' as it was done for showing the existence of B). Therefore we may assume otherwise, whence from Theorem 2.1 it follows that $|\sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} B'_i| \geq \min\{\frac{m}{b}, |S'_b| - \lfloor \frac{m}{2} \rfloor + 1\} = \frac{m}{b}$. Thus applying Proposition 2.2 to B'', it follows that there exists a $\lfloor \frac{m}{2} \rfloor$-set partition $B''' = B''_1, \ldots, B''_{\lfloor \frac{m}{2} \rfloor}$ of a subsequence S''_b of S'_b with $|S''_b| \leq \lfloor \frac{m}{2} \rfloor + \frac{m}{b} - 1$, such that $|\sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} B''_i| = \frac{m}{b}$. Consequently, as in the previous paragraph, it follows that there are at least $\binom{n - (\lfloor \frac{m}{2} \rfloor + \frac{m}{b} - 1) - (b-2)}{\lceil \frac{m}{2} \rceil} \geq \binom{n-m}{\lceil \frac{m}{2} \rceil}$ m-term zero-sum subsequences. Thus, since $n \leq 3m + \lceil \frac{2m-1}{3} \rceil - 2$, it follows in view of Lemma 3.1 that the proof is complete. So we may assume that $n \geq 3m + \lceil \frac{2m-1}{3} \rceil - 1$.

Step 4 (S essentially trichromatic): Suppose that every term of S, with at most $\lfloor \frac{m-4}{3} \rfloor$ exceptions, is equal to one of three elements x, y, $z \in G$. Let n_x, n_y, n_z be the respective multiplicities of x, y and z in S, and w.l.o.g. assume $n_x \geq n_y \geq n_z$. Let $l \leq \lfloor \frac{m-4}{3} \rfloor$ be the number of terms t of S with $t \notin \{x, y, z\}$. In view of steps 2 and 3, it follows for $w \in \{x, y, z\}$ that there are at least $\lfloor \frac{2m-1}{3} \rfloor - \lfloor \frac{m-4}{3} \rfloor \geq \lfloor \frac{m-4}{3} \rfloor + 2 \geq l + 2$ terms of S equal to w.

Claim 1. We proceed to show that if $n_x \leq \lfloor \frac{n}{2} \rfloor - l$, then for each $w \in \{x, y, z\}$ there exists an $\lceil \frac{n}{2} \rceil$-set partition $A^{(w)} = A_1, \ldots, A_{\lceil \frac{n}{2} \rceil}$ of S into cardinality at most two sets, such that if either $t \in A_j$ with $t \notin \{x, y, z\}$, or if $|A_j| = 1$, then $w \in A_j$. Since $n_w \geq l + 2$, then for i with $\lfloor \frac{n}{2} \rfloor - l + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, let $A_i = \{w, t_i\}$, where the t_i are the terms with $t_i \notin \{x, y, z\}$, and if n is odd, then let $A_{\lceil \frac{n}{2} \rceil} = \{w\}$. Let S' be the subsequence of S obtained by deleting all terms contained in the A_i with $i \geq \lfloor \frac{n}{2} \rfloor - l + 1$. To show the claim it suffices to show S' has an $(\lfloor \frac{n}{2} \rfloor - l)$-set partition with all sets of cardinality at most two. However, in view of Proposition 2.1, this will be the case provided no term of S' has multiplicity at least $\lceil \frac{n}{2} \rceil - l + 1$, which we have by assumption of Claim 1. Thus the claim is established.

Claim 2. Next, we proceed to show that if $n_x \geq \lfloor \frac{n}{2} \rfloor - l + 1$, then for each $w \in \{y, z\}$ there exists an $\lceil \frac{n}{2} \rceil$-set partition $A^{(w)} = A_1, \ldots, A_n$ of S into cardinality at most two sets, such that either $x \in A_j$ or $w \in A_j$ for all j, such that if $|A_j| = 1$, then $A_j = \{w\}$, and such that $A_j \neq \{y, z\}$ for all j. Let w' be the remaining element in $\{y, z\} \setminus \{w\}$. Rearrange the sequence S so that all the terms equal to x proceed all the terms equal w, which proceed all the terms equal to w', which proceed all the terms t with $t \notin \{x, y, z\}$, and let $x_1, \ldots, x_n$ be the resulting sequence. Let $A_i = \{x_i, x_{i+\lceil \frac{n}{2} \rceil}\}$ for $i \leq \frac{n}{2}$, and if n is odd, then let $A_{\lceil \frac{n}{2} \rceil} = \{x_{\lceil \frac{n}{2} \rceil}\}$. In view of Step 1 it follows that $n_x \leq \lceil \frac{n}{2} \rceil - 1$. Hence, since $n_w \geq \lfloor \frac{m-4}{3} \rfloor + 2 \geq l + 2$, and since $n_x \geq \lfloor \frac{n}{2} \rfloor - l + 1$, then it follows that the set partition $A^{(w)} = A_1, \ldots, A_{\lceil \frac{n}{2} \rceil}$ satisfies the claim.

Let $A^{(w)} = A_1, \ldots, A_{\lceil \frac{n}{2} \rceil}$ be the respective $\lceil \frac{n}{2} \rceil$-set partition constructed using w from Claim 1 (if $n_x \leq \lfloor \frac{n}{2} \rfloor - l$) or from Claim 2 (if $n_x \geq \lfloor \frac{n}{2} \rfloor - l + 1$), and w.l.o.g. re-index $A^{(w)}$ such that if n is odd, then $|A_{\lceil \frac{n}{2} \rceil}| = 1$, and such that $A_j \subsetneq \{x, y, z\}$ holds precisely for j satisfying $\lfloor \frac{n}{2} \rfloor - l + 1 \leq j \leq \lfloor \frac{n}{2} \rfloor$.

If $n_x \leq \lfloor \frac{n}{2} \rfloor - l$, then suppose for some $w \in \{x, y, z\}$ that difference of elements in $\{x, y, z\} \setminus \{w\}$ generates a subgroup H_b of index $b \leq 2$, and if $n_x \geq \lfloor \frac{n}{2} \rfloor - l + 1$, then suppose that for some $w \in \{y, z\}$ that difference of elements in $\{x, y, z\} \setminus \{w\}$ generates a subgroup H_b of index $b \leq 2$. Let $A_{i_1}, \ldots, A_{i_m}$ be a selection of m sets A_i from $A^{(w)}$.

First suppose that $b = 1$. As seen in Step 2, it is sufficient to show that any such selection $A_{i_1}, \ldots, A_{i_m}$ is good. We proceed to show this claim. If $|\sum_{j=1}^{m} A_{i_j}| \geq m$, then the selection is good in view of Proposition 2.2. Therefore we may assume that $|\sum_{j=1}^{m} A_{i_j}| < m$, whence from Kneser's theorem, it follows that $\sum\limits_{j=1}^{m} A_{i_j}$ is maximally H_a-periodic for some proper, nontrivial subgroup H_a of index a, and that $|A_{i_j}| > |\phi_a(A_{i_j})|$ holds for at least $m-1-(a-2)$ sets A_{i_j}. Hence, since there are at most $\lfloor \frac{m-1}{3} \rfloor < m-a+1$ sets A_i with either $|A_i| = 1$ or $A_i \subsetneq \{x, y, z\}$, it follows that $|A_{i_{j'}}| < |\phi_a(A_{i_{j'}})|$ holds for some $A_{i_{j'}}$ with $A_{i_{j'}} \subseteq \{x, y, z\}$ and $|A_{i_{j'}}| = 2$. Hence, since the difference of the pair from $\{x, y, z\}$ not containing w generates G, it follows that $w \in A_{i_{j'}}$. Thus it follows from the pigeonhole principle and the definition of $A^{(w)}$ that every set A_{i_j} will contain a representative from the common H_a-coset $w + H_a$ (the representative being either w or the other element from $A_{i_{j'}}$, which under the case of Claim 2 will be x). If n is odd, then let $A_{i_k} = A_{\lceil \frac{n}{2} \rceil}$. Otherwise, since there are at least $m-a+1 \geq \frac{m}{a}$ sets A_{i_j} with $|A_{i_j}| > |\phi_a(A_{i_j})| = 1$, then it follows, in view of Proposition 2.2 applied to these $\frac{m}{a}$ sets, that there is a set A_{i_k} with $|A_{i_k}| > |\phi_a(A_{i_k})|$ such that $\sum\limits_{\substack{j=1\\ j\neq k}}^{m} A_{i_j} = \sum\limits_{j=1}^{m} A_{i_j}$. Thus since A_{i_k} is a subset of the H_a-coset $w + H_a$, since every set A_{i_j} contains a representative from the common H_a-coset $w + H_a$, and since $\sum\limits_{j=1}^{m} A_{i_j}$ is H_a-periodic, it follows that $0 \in H_a = mw + H_a \subseteq t + \sum\limits_{\substack{j=1\\ j\neq k}}^{m} A_{i_j}$ for every $t \in A_{i_k}$, whence the selection is good. So we may assume that $b = 2$, and consequently from the definition of $A^{(w)}$ that the difference of elements from every set A_i with $A_i \subseteq \{x, y, z\}$ generates a proper subgroup.

If $|\sum\limits_{j=1}^{m} A_{i_j}| \geq m$, then as seen in the previous paragraph, it follows that the selection $A_{i_1}, \ldots, A_{i_m}$ is good. If this is not the case, then in view of Kneser's Theorem it follows that $\sum\limits_{j=1}^{m} A_{i_j}$ is maximally H_a-periodic with H_a a nontrivial, proper subgroup of index a. Also, if there is a set $A_{i_j} \subseteq \{x, y, z\}$ with $w \in A_{i_j}$ and $|A_{i_j}| > |\phi_a(A_{i_j})|$, then, as in the previous paragraph, it follows that every set A_{i_j} will contain a representative from the common H_a-coset $w + H_a$ implying that the selection $A_{i_1}, \ldots, A_{i_m}$ is again good. Hence if a selection is not good, then all sets A_{i_j} with $|\phi_a(A_{i_j})| = 1$ must satisfy one of the following conditions: (a) $|A_{i_j}| = 1$, or (b) $A_{i_j} \subsetneq \{x, y, z\}$, or (c) $A_{i_j} =$

$\{x,y,z\}\setminus\{w\}$. Since $|\sum_{j=1}^{m} A_{i_j}| < m$, then from Kneser's Theorem it follows that there can be at most $a-2$ sets A_{i_j} with $|\phi_a(A_{i_j})| = 2$, and consequently, in view of the previous sentence, at most $a-2$ sets A_{i_j} with $A_{i_j} \subseteq \{x,y,z\}$, $|A_{i_j}| = 2$, and $w \in A_{i_j}$.

Since there are at most $\lfloor \frac{m-1}{3} \rfloor < m-a+2$ sets A_i satisfying (a) or (b), and since there are at least $m-a+2$ sets A_{i_j} with $|\phi_a(A_{i_j})| = 1$, it follows that there must be at least one set A_{i_j} that is contained in an H_a-coset and that satisfies (c). Hence $|\phi_a(\{x,y,z\}\setminus\{w\})| = 1$, implying that subgroup H_b generated by the difference of elements in $\{x,y,z\}\setminus\{w\}$ is a subgroup of H_a. Hence, since H_a is a proper subgroup, and since H_b has index $b=2$, it follows that $H_b = H_a$. Consequently, as noted in the previous paragraph, it follows that there can be at most $a-2 = b-2 = 0$ sets A_{i_j} with $A_{i_j} \subseteq \{x,y,z\}$, $|A_{i_j}| = 2$, and $w \in A_{i_j}$.

Since $n_w \geq l+2$, it follows that there exists a subset $A_k \subseteq \{x,y,z\}$ with $w \in A_k$ and $|A_k| = 2$. In view of the previous paragraph, any selection $A_{i_1},\ldots,A_{i_m}$ that includes the set A_k will be a good selection. Thus there are at least, in case n even, $2\binom{\lfloor \frac{n}{2}\rfloor -1}{m-1} = \binom{\lfloor \frac{n}{2}\rfloor -1}{m-1} + \binom{\lceil \frac{n}{2}\rceil -1}{m-1}$, and in case n odd, $2\binom{\lfloor \frac{n}{2}\rfloor -1}{m-1} + \binom{\lfloor \frac{n}{2}\rfloor -1}{m-2} = \binom{\lfloor \frac{n}{2}\rfloor -1}{m-1} + \binom{\lceil \frac{n}{2}\rceil -1}{m-1}$, m-term zero-sum subsequences that use one of the two terms contained in A_k. Hence by induction hypothesis it follows that there are at least

$$\binom{\lfloor \frac{n}{2}\rfloor -1}{m-1} + \binom{\lceil \frac{n}{2}\rceil -1}{m-1} + \min\left\{\binom{\lfloor \frac{n}{2}\rfloor -1}{m} + \binom{\lceil \frac{n}{2}\rceil -1}{m}, \binom{n-m-2}{\lceil \frac{2m-1}{3}\rceil}\right\} \tag{9}$$

m-term zero-sum subsequences. In view of the Pascal identity, it follows that $\binom{\lfloor \frac{n}{2}\rfloor -1}{m-1} + \binom{\lceil \frac{n}{2}\rceil -1}{m-1} + \binom{\lfloor \frac{n}{2}\rfloor -1}{m} + \binom{\lceil \frac{n}{2}\rceil -1}{m} = \binom{\lfloor \frac{n}{2}\rfloor}{m} + \binom{\lceil \frac{n}{2}\rceil}{m}$. Hence in view of (9), it follows that the proof will be complete unless

$$\binom{n-m-2}{\lceil \frac{2m-1}{3}\rceil} < \binom{\lceil \frac{n}{2}\rceil -1}{m} + \binom{\lfloor \frac{n}{2}\rfloor -1}{m}, \tag{10}$$

and

$$\binom{\lceil \frac{n}{2}\rceil -1}{m-1} + \binom{\lfloor \frac{n}{2}\rfloor -1}{m-1} + \binom{n-m-2}{\lceil \frac{2m-1}{3}\rceil} < \binom{n-m}{\lceil \frac{2m-1}{3}\rceil}.$$

From the above inequality and the Pascal identity, it follows that

$$\binom{\lceil\frac{n}{2}\rceil-1}{m-1}+\binom{\lfloor\frac{n}{2}\rfloor-1}{m-1}<\binom{n-m-2}{\lceil\frac{2m-1}{3}\rceil-1}+\binom{n-m-1}{\lceil\frac{2m-1}{3}\rceil-1}. \tag{11}$$

From (10) it follows that $n \geq 2m+2$. Hence applying to (11) the binomial identity given in (3), as well as the binomial identity $\binom{n}{m}=\frac{n}{n-m}\binom{n-1}{m}$, it follows that

$$\binom{\lceil\frac{n}{2}\rceil-1}{m-1}+\binom{\lfloor\frac{n}{2}\rfloor-1}{m-1}<\frac{\lceil\frac{2m-1}{3}\rceil}{(n-m-\lceil\frac{2m-1}{3}\rceil-1)}(1+\frac{n-m-1}{n-m-\lceil\frac{2m-1}{3}\rceil})\binom{n-m-2}{\lceil\frac{2m-1}{3}\rceil}.$$

Applying (3) to the above inequality yields

$$\binom{\lceil\frac{n}{2}\rceil-1}{m}+\binom{\lfloor\frac{n}{2}\rfloor-1}{m}<\frac{\lceil\frac{n}{2}\rceil-m}{m}\cdot\frac{\lceil\frac{2m-1}{3}\rceil}{(n-m-\lceil\frac{2m-1}{3}\rceil-1)}(1+\frac{n-m-1}{n-m-\lceil\frac{2m-1}{3}\rceil})\binom{n-m-2}{\lceil\frac{2m-1}{3}\rceil}.$$

Hence from (10) it follows that

$$1<\frac{\frac{n+1}{2}-m}{m}\cdot\frac{\frac{2m+1}{3}}{(n-m-\frac{2m+1}{3}-1)}(1+\frac{n-m-1}{n-m-\frac{2m+1}{3}}),$$

implying that $3(m-1)n^2-(10m^2+7m+1)n+9m^3+17m^2+8m+2<0$. Hence from the quadratic formula, it follows that $8m^4-44m^3-177m^2-86m-25\leq 0$, else the square root of the discriminant will be imaginary. However, from Theorem 2.3 it follows that the roots of $8m^4-44m^3-177m^2-86m-25$ are bounded from above by 9, whence $8m^4-44m^3-177m^2-86m-25>0$ holds for $m>10$, a contradiction. So we may assume that if $n_x \leq \lfloor\frac{n}{2}\rfloor-l$, then none of $x-z$, $x-y$, and $y-z$ generates a subgroup H_b of index $b\leq 2$, and if $n_x\geq\lfloor\frac{n}{2}\rfloor-l+1$, then none of $x-y$ and $x-z$ generates a subgroup H_b of index $b\leq 2$ in G.

For $t\in\{x,y,z\}$ if $n_x\leq\lfloor\frac{n}{2}\rfloor-l$, and for $t\in\{y,z\}$ if $n_x\geq\lfloor\frac{n}{2}\rfloor-l+1$, let H_{b_t} be the subgroup of index b_t generated by the difference of the elements in $\{x,y,z\}\setminus\{t\}$. From the conclusion of the previous paragraph, it follows that $b_t>2$ for each t. Thus given any selection $A_{i_1},\ldots,A_{i_m}$ with all A_{i_j} satisfying $A_{i_j}\subseteq\{x,y,z\}$ and $|A_{i_j}|=2$, it follows from the pigeonhole principle that there are at least $\frac{m}{b_t}-1$ sets A_{i_j} equal to $\{x,y,z\}\setminus\{t\}$ for some t. Note that $\sum\limits_{i=1}^{\frac{m}{b_t}-1}\{x,y,z\}\setminus\{t\}$ is an H_{b_t}-coset, implying that $\sum\limits_{j=1}^{m}A_{i_j}$ is maximally H_a-periodic with $H_{b_t}\leq H_a$. Thus in view of Proposition 2.2 applied with elements considered modulo H_{b_t}, it follows that there exists a re-indexing such that

$$|\sum_{j=1}^{\frac{m}{b_t}-1+b_t-1} A_{i_j}| = |\sum_{j=1}^{m} A_{i_j}|; \tag{12}$$

furthermore, from Kneser's Theorem it follows that $|\phi_a(A_{i_j})|=1$ for $i>\frac{m}{b_t}+b_t-2$, since otherwise $\left|\phi_a\left(\sum\limits_{j=1}^{\frac{m}{b_t}+b_t-2}A_{i_j}\right)\right| < \left|\phi_a\left(\sum\limits_{j=1}^{\frac{m}{b_t}+b_t-2}A_{i_j}\right)+\phi_a\left(\sum\limits_{j=\frac{m}{b_t}+b_t-1}^{m}A_{i_j}\right)\right|$, implying

$$\left|\sum_{j=1}^{\frac{m}{b_t}+b_t-2}A_{i_j}\right| < \left|\left(\sum_{j=1}^{\frac{m}{b_t}+b_t-2}A_{i_j}\right)+\sum_{j=\frac{m}{b_t}+b_t-1}^{m}A_{i_j}\right| = |\sum_{j=1}^{m}A_{i_j}|,$$

which contradicts (12). Since $A_{i_j}\subseteq\{x,y,z\}$ with $|A_{i_j}|=2$ holds for all j, and since $|\phi_{b_t}(\{x,y,z\}\setminus\{t\})|=1$ implies $|\phi_a(\{x,y,z\}\setminus\{t\})|=1$ (since $H_{b_t}\leq H_a$), it follows in view of the pigeonhole principle that every set A_{i_j} contains a representative from the H_a-coset $\{x,y,z\}\setminus\{t\}+H_a$, whence from (12) and the previous sentence it follows that there are at least $2^{m-\frac{m}{b_t}-b_t+2}>0$ ways to select a term from each A_{i_j} and have the resulting m-term sequence be zero-sum. Thus we conclude that there are at least $2^{m-\frac{m}{b_t}-b_t+2}\binom{\lfloor\frac{n}{2}\rfloor-\lfloor\frac{m-4}{3}\rfloor}{m}$ m-term zero-sum subsequences. If $b_t\neq\frac{m}{2}$ for every such selection $A_{i_1},\ldots,A_{i_m}$, then in view of $b_t>2$, it follows for $m\geq 30$ that $2^{m-\frac{m}{b_t}-b_t+2}\geq 2^{\frac{2}{3}m-1}=2\cdot 4^{\frac{m}{3}-1}\geq 2\cdot 3^{\lfloor\frac{m-1}{3}\rfloor}$, whence the proof is complete in view of Lemma 3.3 and Step 3. So we may assume $b_t=\frac{m}{2}$ for some such selection $A_{i_1},\ldots,A_{i_m}$; and it suffices to further show that each selection $A_{i_1},\ldots,A_{i_m}$, with all A_{i_j} satisfying $A_{i_j}\subseteq\{x,y,z\}$ and $|A_{i_j}|=2$, and with $b_t=\frac{m}{2}$, also has at least $2\cdot 3^{\lfloor\frac{m-1}{3}\rfloor}$ ways to select an m-term zero-sum subsequence. We proceed to show this, which will complete the proof of Step 4.

Since $b_t=\frac{m}{2}$, we may w.l.o.g. by translation assume $\{x,y,z\}\setminus\{t\}=\{0,s\}$, where s has order 2. Since $t-0=t$ does not generate a subgroup with index $b\leq 2$, implying the order of t is strictly less than $\frac{m}{2}$, and since $|G/H_{b_t}| = \frac{m}{2}$, it follows that $\phi_{b_t}(t)$ generates a proper subgroup $H_{b'}$ of G/H_{b_t} with index $b' \geq 2$ in G/H_{b_t}.

Suppose there are at least $2\lfloor\frac{m-1}{3}\rfloor + 2$ sets A_{i_j} with $|\phi_{b_t}(A_{i_j})| = 1$. Then, since $|H_{b_t}| = 2$ implies that $|A_{i_{j_1}} + A_{i_{j_2}}| = |A_{i_{j_1}}|$ when $|\phi_{b_t}(A_{i_{j_1}})| = |\phi_{b_t}(A_{i_{j_2}})| = 1$, it follows that we can re-index such that $|\sum\limits_{j=1}^{m-(2\lfloor\frac{m-1}{3}\rfloor+1)} A_{i_j}| = |\sum\limits_{j=1}^{m} A_{i_j}|$, with $|\phi_{b_t}(A_{i_j})| = 1$ for $j > m - (2\lfloor\frac{m-1}{3}\rfloor + 1)$. Since there are at least $2^{m-\frac{m}{b_t}-b_t+2} > 0$ ways to select an m-term zero-sum from the selection $A_{i_1}, \ldots, A_{i_m}$, it follows that $0 \in \sum\limits_{j=1}^{m} A_{i_j}$. Thus, since $|\sum\limits_{j=1}^{m-(2\lfloor\frac{m-1}{3}\rfloor+1)} A_{i_j}| = |\sum\limits_{j=1}^{m} A_{i_j}|$, then it follows that there will be at least $2^{2\lfloor\frac{m-1}{3}\rfloor+1} \geq 2 \cdot 3^{\lfloor\frac{m-1}{3}\rfloor}$ ways to select an m-term zero-sum subsequence from the selection $A_{i_1}, \ldots, A_{i_m}$, completing the proof as noted earlier. So, we may assume there are at least $m - (2\lfloor\frac{m-1}{3}\rfloor + 1) \geq \lceil\frac{m-1}{3}\rceil \geq \frac{m}{2b'} - 1 = \frac{b_t}{b'} - 1$ sets A_{i_j} with $|\phi_{b_t}(A_{i_j})| = 2$.

Hence, since $|\phi_{b_t}(\{0, s\})| = 1$, and since $\phi_{b_t}(\{0, t\}) = \phi_{b_t}(\{s, t\})$ (since $|\phi_{b_t}(\{0, s\})| = 1$ implies $\phi_{b_t}(0) = \phi_{b_t}(s)$), it follows that there are at least $\frac{b_t}{b'} - 1$ sets A_{i_j} that modulo H_{b_t} have the difference of their elements generating the subgroup $H_{b'} = \langle\phi_{b_t}(t)\rangle$. Note that $\sum\limits_{i=1}^{\frac{b_t}{b'}-1} \phi_{b_t}(\{0, t\}) = H_{b'}$. Hence, since there at least $\frac{b_t}{b'} - 1$ sets A_{i_j} with $|\phi_{b_t}(A_{i_j})| = 2$, and since there are at least $\frac{m}{b_t} - 1$ sets A_{i_j} equal to $\{x, y, z\} \setminus \{t\} = \{0, s\}$, it follows in view of Proposition 2.2 applied with elements considered in $(G/H_{b_t})/H_{b'}$, that there exists a re-indexing such that $|\sum\limits_{j=1}^{\frac{m}{b_t}-1+\frac{b_t}{b'}-1+b'-1} A_{i_j}| = |\sum\limits_{j=1}^{m} A_{i_j}|$; furthermore, from Kneser's Theorem it follows that $|\phi_a(A_{i_j})| = 1$ for $i > \frac{m}{b_t} - 1 + \frac{b_t}{b'} - 1 + b' - 1 = \frac{b_t}{b'} + b' - 1$, since otherwise

$$|\sum_{j=1}^{\frac{b_t}{b'}+b'-1} A_{i_j}| < \left|\left(\sum_{j=1}^{\frac{b_t}{b'}+b'-1} A_{i_j}\right) + \sum_{j=\frac{b_t}{b'}+b'}^{m} A_{i_j}\right|$$

will hold, a contradiction. Thus, since $\frac{b_t}{b'} + b' - 1 \leq \frac{m}{4} + 1$, and since $0 \in \sum\limits_{j=1}^{m} A_{i_j}$, it follows that there will be at least $2^{\frac{3}{4}m-1} \geq 2 \cdot 3^{\lfloor\frac{m-1}{3}\rfloor}$ ways to select an m-term zero-sum subsequence from the selection $A_{i_1}, \ldots, A_{i_m}$, completing the proof of Step 4 as noted earlier. So we may assume that given any $x, y, z \in G$, there are at least $\lfloor \frac{m-1}{3} \rfloor$ terms of S not equal to x or y or z.

Step 5 (The general case:) In view of Steps 1, 2, 3, and 4, and Proposition 2.1, it follows that there exists an $(n-m+1)$-set partition $P = P_1, \ldots, P_{n-m+1}$ of S with $|P_i| = 1$ for $i > \lceil \frac{m-1}{3} \rceil$. Let $P' = P_1, \ldots, P_{\lceil \frac{m-1}{3} \rceil}$, and let S' be the corresponding subsequence partitioned by the set partition P'. Apply Theorem 2.1 to the subsequence S' of S with $\lceil \frac{m-1}{3} \rceil$-set partition P', and let S'' be the resulting subsequence, H_a the resulting subgroup of index a, and $A = A_1, \ldots, A_{\lceil \frac{m-1}{3} \rceil}$ the resulting set partition of S''.

Suppose that $|\sum_{i=1}^{\lceil \frac{m-1}{3} \rceil} A_i| \geq m = \sum_{i=1}^{\lceil \frac{m-1}{3} \rceil} |A_i| - \lceil \frac{m-1}{3} \rceil + 1$. Then applying Theorem 2.2 to A and S'', it follows that there exists a subsequence T of S'' of length at most m with a set partition $B = B_1, \ldots, B_{\lceil \frac{m-1}{3} \rceil}$ such that $|\sum_{i=1}^{\lceil \frac{m-1}{3} \rceil} B_i| \geq m$. Then given any subsequence T' of $S \setminus T$ of length $m - \lceil \frac{m-1}{3} \rceil = \lceil \frac{2m-1}{3} \rceil$, we can find a selection of $\lceil \frac{m-1}{3} \rceil$ terms from T, one from each of the $B_1, \ldots, B_{\lceil \frac{m-1}{3} \rceil}$, that sum to the additive inverse of the sum of the terms from the $\lceil \frac{2m-1}{3} \rceil$-term subsequence T'. Consequently, there will be least $\binom{n-m}{\lceil \frac{2m-1}{3} \rceil}$ m-term zero-sum subsequences, completing the proof. So we can assume that

$$|\sum_{i=1}^{\lceil \frac{m-1}{3} \rceil} A_i| < m = \sum_{i=1}^{\lceil \frac{m-1}{3} \rceil} |A_i| - \lceil \frac{m-1}{3} \rceil + 1. \tag{13}$$

Hence from Theorem 2.1 it follows that H_a is a proper, nontrivial subgroup, and that either $N(A, H_a) = 1$ and $E(A, H_a) \leq a - 2$, or else $N(A, H_a) = 2$, $|H_a| = 2$, and $E(A, H_a) \leq \frac{m}{2} - \lceil \frac{m-1}{3} \rceil - 2 \leq \lfloor \frac{m-10}{6} \rfloor$. The case $N(A, H_a) = 1$ and $E(A, H_a) \leq a - 2$ can be handled by a minor modification of the arguments from the third paragraph of Step 3 (simply replace $\lfloor \frac{m}{2} \rfloor$ by $\lceil \frac{m-1}{3} \rceil$ where appropriate). Therefore we may assume the latter case holds.

Since $N(A, H_a) = 2$, choose x, $y \in G$ so that $\phi_a(x)$, $\phi_a(y) \in G/H_a$ are the two elements from $\phi_a\left(\bigcap_{i=1}^{\lceil \frac{m-1}{3} \rceil} (A_i + H_a)\right)$. Suppose first that $\phi_a(x - y)$ generates a proper subgroup $H_{a'}/H_a$ of

G/H_a. If there does not exist a set $A_{j'}$ such that $(\{x,y\}+H_a) \subseteq A_{j'}$, then there will be at least $\lceil \frac{m-1}{3} \rceil = \lceil \frac{m-1}{3} \rceil (|H_a|-1)$ holes contained among the sets A_{i_j}, which in view of the comments after Theorem 2.1 implies that (13) cannot hold, a contradiction. Therefore we may assume that there exists a set $A_{j'}$ with $(\{x,y\}+H_a) \subseteq A_{j'}$.

For $i = j'$, let $B_{j'} = (\{x,y\}+H_a) \cap A_{j'} = \{x,y\}+H_a$, and for $i \neq j'$, let B_i be a cardinality two subset of $A_i \cap (\{x,y\}+H_a)$ with $|\phi_a(B_i)| = 2$. Then, since $\phi_a(x-y)$ generates a proper subgroup $H_{a'}/H_a$, and since $\lceil \frac{m-1}{3} \rceil \geq \frac{m}{4} \geq |G/H_{a'}|$, it follows that $\sum\limits_{i=1}^{\lceil \frac{m-1}{3} \rceil} B_i$ is an $H_{a'}$-coset. Observe that all but at most $E(A, H_a) \leq \frac{m}{2} - \lceil \frac{m-1}{3} \rceil - 2$ terms of S are from the same $H_{a'}$-coset $x + H_{a'}$. Let T be the subsequence of S partitioned by the set partition $B = B_1, \ldots, B_{\lceil \frac{m-1}{3} \rceil}$. Hence, since $B_i \subseteq x + H_{a'}$ for all i, and since $\sum\limits_{i=1}^{\lceil \frac{m-1}{3} \rceil} B_i$ is an $H_{a'}$-coset, it follows that given any $\lceil \frac{2m-1}{3} \rceil$-term subsequence T' of $S \setminus T$ with all terms from the coset $x + H_{a'}$, then we can find a selection of $\lceil \frac{m-1}{3} \rceil$ terms from T, one from each $B_1, \ldots, B_{\lceil \frac{m-1}{3} \rceil}$, that sums to the additive inverse of the sum of terms from T'. Hence, since there are at least $n - (2\lceil \frac{m-1}{3} \rceil + 2 + E(A', H_a)) \geq n - (2\lceil \frac{m-1}{3} \rceil + 2 + \frac{m}{2} - \lceil \frac{m-1}{3} \rceil - 2) \geq n - m$ terms of $S \setminus T$ from the coset $x + H_{a'}$, it follows that there are at least $\binom{n-m}{\lceil \frac{2m-1}{3} \rceil}$ m-term zero-sum subsequences, completing the proof. So we may assume that $\phi_a(x-y)$ generates G/H_a.

Let x' be the other element from the coset $x + H_a$, and let y' be the other element from the coset $y + H_a$. Let n_x, $n_{x'}$, n_y, and $n_{y'}$ be the respective multiplicities of x, x', y, and y' in S. Since, as noted previously, there is a set $A_{j'}$ such that $(\{x,y\}+H_a) \subseteq A_{j'}$, it follows that $n_x \geq 1$, $n_{x'} \geq 1$, $n_y \geq 1$, and $n_{y'} \geq 1$. We may w.l.o.g. assume that $n_x + n_{x'} \geq n_y + n_{y'}$, that $n_x \geq n_{x'}$, and that $n_y \geq n_{y'}$. Remove two terms from S, one equal to x and one equal to x', and let the resulting sequence be T. Let B_0 be the set consisting of the two removed terms. Rearrange the terms of T so that all terms equal to x proceed all terms equal to x', which proceed all terms equal to y, which proceed all terms equal to y', which proceed all terms t with $t \notin \{x, x', y, y'\}$, and let $x_1, \ldots, x_{n-2}$

be the resulting sequence. Let $B_i = \{x_i, x_i + \lceil \frac{n}{2} \rceil - 1\}$ for $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor - 1$, and, in case n odd, let $B_{\lceil \frac{n}{2} \rceil - 1} = \{x_{\lceil \frac{n}{2} \rceil - 1}\}$. In view of Step 1, it follows that $B = B_1, \ldots, B_{\lceil \frac{n}{2} \rceil - 1}$ is an $(\lceil \frac{n}{2} \rceil - 1)$-set partition of T. As seen in the ninth paragraph of Step 4, it suffices by induction hypothesis to show that any selection $B_0, B_{i_1}, \ldots, B_{i_{m-1}}$ containing B_0 is good. We proceed to show this.

If $|\sum_{j=1}^{m-1} \phi_a(B_{i_j})| \geq \frac{m}{2}$, then, since B_0 is an H_a-coset, it follows that $|B_0 + \sum_{j=1}^{m-1} B_{i_j}| \geq m$, whence from Proposition 2.2 it follows that the selection $B_0, B_{i_1}, \ldots, B_{i_{m-1}}$ is good. Hence we may assume that

$$|\sum_{j=1}^{m-1} \phi_a(B_{i_j})| < \frac{m}{2}. \tag{14}$$

Suppose that $n_x + n_{x'} > \lceil \frac{n}{2} \rceil$. Then every set B_{i_j} will contain a representative from the common H_a-coset $x + H_a$. Hence, since B_0 is H_a-periodic, it follows that $0 \in B_0 + \sum_{j=1}^{m-1} B_{i_j}$.

Suppose further that $|\phi_a(B_{i_k})| = 1$ holds for some B_{i_k} with $i_k \geq 1$. Then $B_0 + \sum_{\substack{j=1 \\ j \neq k}}^{m-1} B_{i_j} = B_0 + \sum_{j=1}^{m-1} B_{i_j}$, and it follows that either $|B_{i_k}| = 1$, or else there will be at least two ways to represent every $x \in B_0 + \sum_{j=1}^{m-1} B_j$. Hence, since $0 \in B_0 + \sum_{j=1}^{m-1} B_j$, it follows that the selection $B_0, B_{i_1}, \ldots, B_{i_{m-1}}$ is good, completing the proof as noted earlier. So we may assume $|\phi_a(B_{i_k})| = 2$ for all $i_k \geq 1$.

Since $|\phi_a(B_{i_k})| = 2$ for all $i_k \geq 1$, and since $|\phi_a(\{x, x'\})| = 1$, it follows that there does not exist a set B_{i_j} with $i_j \geq 1$ and $B_{i_j} = \{x, x'\}$. Hence, since there are at most $E(A, H_a) \leq \frac{m-10}{6}$ terms t with $t \notin \{x, x', y, y'\}$, and since every set B_{i_j} contains either x or x', it follows that there are at least $m - 2 - \frac{m-10}{6} \geq \frac{m}{2}$ sets B_{i_j} with the difference of terms in B_{i_j} equal modulo H_a to $\phi_a(x - y)$. Thus, since $\phi_a(x - y)$ generates G/H_a, it follows that (14) cannot hold, a contradiction. So we may assume that $n_x + n_{x'} \leq \lceil \frac{n}{2} \rceil$.

Since $n_x + n_{x'} \leq \lceil \frac{n}{2} \rceil$, since $n_x + n_{x'} \geq n_y + n_{y'}$, and since all but at most $E(A', H_a) \leq \frac{m}{2} - \lceil \frac{m-1}{3} \rceil - 2 \leq \lfloor \frac{m-10}{6} \rfloor$ terms of S are equal to one of x, x', y, or y', it follows that at least

$(m-3) - \frac{m-10}{6} \geq \frac{m}{2}$ sets B_{i_j} have $\phi_a(B_{i_j}) = \{\phi_a(x), \phi_a(y)\}$. Hence, since $\phi_a(x) - \phi_a(y)$ generates G/H_a, it follows that $|\sum_{j=1}^{m-1} \phi_a(B_{i_j})| \geq \frac{m}{2}$, contradicting (14) again, and completing the proof. $\square$

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