Monochromatic and zero-sum sets of nondecreasing modified-diameter

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Abstract

Let m be a positive integer whose smallest prime divisor is denoted by p, and let $\mathbb{Z}_m$ denote the cyclic group of residues modulo m. For a set $B = \{x_1, x_2, \ldots, x_m\}$ of m integers satisfying $x_1 < x_2 < \cdots < x_m$, and an integer j satisfying $2 \leq j \leq m$, define $g_j(B) = x_j - x_1$. Furthermore, define $f_j(m, 2)$ (define $f_j(m, \mathbb{Z}_m)$) to be the least integer N such that for every coloring $\Delta : \{1, \ldots, N\} \to \{0, 1\}$ (every coloring $\Delta : \{1, \ldots, N\} \to \mathbb{Z}_m$), there exist two m-sets $B_1, B_2 \subset \{1, \ldots, N\}$ satisfying: (i) $\max(B_1) < \min(B_2)$, (ii) $g_j(B_1) \leq g_j(B_2)$, and (iii) $|\Delta(B_i)| = 1$ for $i = 1, 2$ (and (iii) $\sum_{x \in B_i} \Delta(x) = 0$ for $i = 1, 2$). We prove that $f_j(m, 2) \leq 5m - 3$ for all j, with equality holding for $j = m$, and that $f_j(m, \mathbb{Z}_m) \leq 8m + \frac{m}{p} - 6$. Moreover, we show that $f_j(m, 2) \geq 4m - 2 + (j-1)k$, where $k = \left\lfloor \left(-1 + \sqrt{\frac{8m-9+j}{j-1}}\right)/2 \right\rfloor$, and, if m is prime or $j \geq \frac{m}{p} + p - 1$, that $f_j(m, \mathbb{Z}_m) \leq 6m - 4$. We conclude by showing $f_{m-1}(m, 2) = f_{m-1}(m, \mathbb{Z}_m)$ for $m \geq 9$.

1 Introduction

Let $[a, b]$ denote the set of integers between a and b, inclusive. For a set S, an S-coloring of $[1, N]$ is a function $\Delta : [1, N] \to S$. If $S = \{0, 1, \ldots, r-1\}$, then we call Δ an r-coloring. The following is the Erdős-Ginzburg-Ziv (EGZ) theorem, [1] [14] [29].

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Theorem 0. *Let m be a positive integer. If $\Delta : [1, 2m-1] \to \mathbb{Z}_m$, then there exist distinct integers $x_1, x_2, \ldots, x_m \in [1, 2m-1]$ such that $\sum_{i=1}^{m} \Delta(x_i) = 0$. Moreover, $2m - 1$ is the smallest number for which the above assertion holds.*

The EGZ theorem can be viewed as a generalization of the pigeonhole principle for 2 boxes (since the m-term zero-sum subsequences of a sequence consisting only of 0's and 1's are exactly the monochromatic m-term subsequences). As such, several theorems of Ramsey-type have been generalized similarly by considering $\mathbb{Z}_m$-colorings and zero-sum configurations rather than 2-colorings and monochromatic configurations. When in such a theorem the size of the configuration needed to guarantee a monochromatic sub-configuration equals the size of the configuration needed to guarantee a zero-sum sub-configuration (as it does for EGZ versus the pigeonhole principle), we say that the theorem zero-sum generalizations. The most well known such theorem is the zero-trees theorem [17] [32]. Two surveys of related results and open problems appear in [3] [12], and some examples of other various extensions of EGZ appear in [10] [11] [16] [18] [19] [20] [21] [26] [30] [31].

One of the first Ramsey-type problems considered with respect to zero-sum generalizations was the nondecreasing diameter problem introduced by Bialostocki, Erdős, and Lefmann [8]. For a set $B = \{x_1, x_2, \ldots, x_m\}$ of m positive integers satisfying $x_1 < x_2 < \cdots < x_m$, and an integer j satisfying $2 \le j \le m$, let $g_j(B) = x_j - x_1$. Note that when $j = m$, then $g_m(B)$ is the diameter of the set B. Let $f_j(m, 2)$ (let $f_j(m, \mathbb{Z}_m)$) be the least integer N such that for every coloring $\Delta : [1, N] \to \{0, 1\}$ (for every coloring $\Delta : [1, N] \to \mathbb{Z}_m$), there exist two m-sets B_1, $B_2 \subset [1, N]$ satisfying (i) $\max(B_1) < \min(B_2)$, (ii) $g_j(B_1) \le g_j(B_2)$, and (iii) $|\Delta(B_i)| = 1$ for $i = 1, 2$ (and (iii) $\sum_{x \in B_i} \Delta(x) = 0$ for $i = 1, 2$). Bialostocki, Erdős, and Lefmann introduced the functions $f_m(m, 2)$ and $f_m(m, \mathbb{Z}_m)$ and showed that $f_m(m, 2) = f_m(m, \mathbb{Z}_m) = 5m - 3$, thus obtaining one of the first 2-color zero-sum generalizations for a Ramsey-type problem [8]. They also introduced a notion of zero-sum generalization for Ramsey-type problems involving arbitrary r-colorings (not just 2-colorings), and showed that the corresponding 3-color version of the nondecreasing diameter problem for two m-sets also zero-sum generalized. Recently, the four color case was shown to zero-sum generalize [23], but the cases with $r > 4$ remain open and difficult.

In this paper we introduce and study the functions $f_j(m, 2)$ and $f_j(m, \mathbb{Z}_m)$ with $j < m$, thus studying the nondecreasing diameter problem by varying the notion of diameter by the parameter j. One of our main tools is a corollary to a recent generalization of results of Mann [28], Olson

[30], Bolobás and Leader [10], and Hamidoune [25], that was developed by the first author [22] while studying the original nondecreasing diameter problem for four colors [23]. However, using the original statement of the corollary from [22] involves a sometimes painful technical obstacle (namely, after application of the theorem the resulting n-set partition may contain more sets of cardinality greater than one than the original n-set partition), easily enough overcome for the original nondecreasing diameter problem [23], but more difficult for more complicated problems. In order to avoid this technical obstacle, we re-derive an augmented version of the corollary. The augmented version, cited in our paper as Theorem 2.7, allows for a (relatively) smoother application of the theorem in practice, as also seen in [24].

For a positive integer m, let $F(m,2) = \max_j\{f_j(m,2)\}$ and let $F(m,\mathbb{Z}_m) = \max_j\{f_j(m,\mathbb{Z}_m)\}$. This project was begun when A. Bialostocki suggested the following two conjectures [2].

Conjecture 1.1.

$$\liminf_{m\to\infty} \frac{F(m,\mathbb{Z}_m)}{F(m,2)} = 1.$$

Conjecture 1.2. *If $j \geq 2$ is an integer which is either fixed or of the form $m-k$ for some fixed positive integer k, then*

$$\liminf_{j\leq m\to\infty} \frac{f_j(m,\mathbb{Z}_m)}{f_j(m,2)} = 1.$$

Among other results, we support Conjecture 1.1, proving that $\liminf_{m\to\infty} \frac{F(m,\mathbb{Z}_m)}{F(m,2)} \leq 1.2$. Furthermore, we prove Conjecture 1.2 for $j = m-1$ by showing that

$$f_{m-1}(m,2) = f_{m-1}(m,\mathbb{Z}_m) = 5m-4 \quad \text{for } m \geq 9.$$

The paper is organized as follows. Section 2 contains definitions, terminology, and results used in Sections 3 and 4, which contain results addressing Conjectures 1.1 and 1.2, respectively.

2 Preliminaries

We recall some theorems from additive number theory, but first we need to introduce terminology used in [22] and [29]. If G is an abelian group and A, $B \subseteq G$, then their *sumset* is $A+B = \{a+b \mid a \in A, b \in B\}$. A set $A \subseteq G$ is said to be *H_a-periodic*, if it is the union of H_a-cosets for some nontrivial subgroup H_a of G, and otherwise, A is called *aperiodic*. We say that A is *maximally*

H_a-periodic, if A is H_a-periodic, and H_a is the maximal subgroup for which A is periodic; in this case, $H_a = \{x \in G \mid x + A = A\}$, and H_a is sometimes referred to as the *stabilizer* of A. We will use $\phi_a : G \to G/H_a$ to denote the natural homomorphism. If S is a sequence of elements from G, then an *n-set partition* of S is a partition of the sequence S into n nonempty subsequences, $A_1, \ldots, A_n$, such that the terms in each subsequence A_i are all distinct (thus allowing each subsequence A_i to be considered a set). A sequence of elements from $\mathbb{Z}_m$ is *zero-sum* if the sum of its terms is zero. An *affine transformation* is any map $\gamma : \mathbb{Z}_m \to \mathbb{Z}_m$ given by $\gamma(x) = kx + b$, where $k, b \in \mathbb{Z}_m$ and $gcd(k, m) = 1$. Furthermore, $|S|$ denotes the cardinality of S, if S is a set, and the length of S, if S is a sequence. If S is an ordered set and r is an integer satisfying $|S| \geq r$, then elements $y_1 < y_2 < \cdots < y_r \in S$ are said to be a *final segment* if $y_i = \max(S \setminus \{y_{i+1}, y_{i+2}, \ldots, y_r\})$ for $i = 1, 2, \ldots, r$. Analogously, integers $y_1 < y_2 < \cdots < y_r \in S$ are said to be an *initial segment* if $y_i = \min(S \setminus \{y_{i-1}, y_{i-2}, \ldots, y_1\})$ for $i = 1, 2, \ldots, r$. Finally, for $j \in \mathbb{Z}_m$, we denote by $\overline{j}$ the least non-negative integer representative of j.

Next, we introduce helpful notation and terminology dealing specifically with our problem. Let S_1 and S_2 be sequences. Then $S_1 \cup S_2$ denotes the concatenation of S_1 with S_2, and if S_2 is a subsequence of S_1, then $S_1 \setminus S_2$ denotes the sequence obtained from S_1 by deleting the terms from S_2. Let $\Delta : S \to C$ be a C-coloring of the set S. The sequence of colors given by Δ will often be abbreviated as a string using exponential notation (e.g. the sequence given by the coloring $\Delta([1,3]) = 1$, $\Delta([4,7]) = 2$ is abbreviated by $1^3 2^4$). If $c \in C$ and $\Delta^{-1}(c) = \{x_1, x_2, \ldots, x_s\}$, where $x_1 < x_2 < \cdots < x_s$, then for an integer $r \leq s$, define

$$\Pi(r, c) = x_r \quad \text{and} \quad \amalg(r, c) = x_{s-r+1}.$$

Let $\Delta : S \to \mathbb{Z}_m$ be a coloring of the set S. A set $B \subset S$ is *zero-sum* if $\sum_{x \in B} \Delta(x) = 0$. Further, Δ is said to *reduce to monochromatic* if either $|\Delta(S)| \leq 2$ or there exists $B \subset S$ such that $|B| \leq m - 1$ and $|\Delta(S \setminus B)| = 1$. Observe that in either case there exists a natural induced coloring $\Delta^* : S \to \{0, 1\}$ such that every m-element monochromatic set under Δ^* is zero-sum under Δ. Finally, let m and j be integers satisfying $2 \leq j \leq m$, and let $\Delta : S \to \{0, 1\}$ (let $\Delta : S \to \mathbb{Z}_m$) be a coloring. Then two m-sets $B_1, B_2 \subset S$ are said to be an *(m, j)-solution* (an *$(m, j, \mathbb{Z}_m)$-solution*) if $\max(B_1) < \min(B_2)$, $g_j(B_1) \leq g_j(B_2)$, and $|\Delta(B_1)| = |\Delta(B_2)| = 1$ (and $\sum_{x \in B_i} \Delta(x) = 0$ for $i = 1, 2$).

First we state a theorem, which is an easy consequence of the Pigeonhole Principle, sometimes referred to as the Caveman Theorem since its roots extend back so far [15].

Theorem 2.1. *Let S be a sequence of elements from a finite abelian group G. If $|S| = |G|$, then there exists a nonempty zero-sum subsequence consisting of consecutive terms of S.*

The following theorem is the Cauchy-Davenport Theorem [29] [13].

Theorem 2.2. *Let m be a prime and let n be a positive integer. If $A_1, A_2, \ldots, A_n$ is a collection of subsets of $\mathbb{Z}_m$, then*

$$\left|\sum_{i=1}^{n} A_i\right| \geq \min\{m, \sum_{i=1}^{n} |A_i| - n + 1\}.$$

Next, we will need the following slightly stronger form of the EGZ theorem [12].

Theorem 2.3. *Let k, m be positive integers such that $k|m$. If $\Delta : [1, m+k-1] \to \mathbb{Z}_m$, then there exist distinct integers $x_1, x_2, \ldots, x_m \in [1, m+k-1]$ such that $\sum_{i=1}^{m} \Delta(x_i) \equiv 0 \mod k$. Moreover, $m + k - 1$ is the smallest number for which the above assertion holds.*

The following theorem turns out to be useful. The proofs of parts (a) and (b) appear in [5] and [9] [7], respectively.

Theorem 2.4. *Let $m \geq 4$ be an integer, and let $\Delta : S \to \mathbb{Z}_m$ be a coloring of a set of integers S for which $|\Delta(S)| \geq 3$.*
(a) If $|S| = 2m - 2$, then there exist distinct integers $x_1, \ldots, x_m$ such that $\sum_{i=1}^{m} \Delta(x_i) = 0$.
(b) If $|S| = 2m - 3$, and there are not distinct integers $x_1, \ldots, x_m$ such that $\sum_{i=1}^{m} \Delta(x_i) = 0$, then $\Delta(S) = \{a, b, c\}$, where $\left|\Delta^{-1}(a)\right| = m - 1$, $\left|\Delta^{-1}(b)\right| = m - 3$, and $\left|\Delta^{-1}(c)\right| = 1$; (c) moreover, up to affine transformation we may assume that $a = 0$, $b = 1$, and $c = 2$.

The following simple proposition will be helpful [7].

Proposition 2.5. *Let S be a sequence of elements from a finite abelian group G, and let $A = A_1, \ldots, A_n$ be an n-set partition of S, where $|\sum\limits_{i=1}^{n} A_i| = r > 1$. Then there exists a subsequence S' of S and an n'-set partition $A' = A_{j_1}, \ldots, A_{j_{n'}}$ of S', which is a subsequence of the n-set partition $A = A_1, \ldots, A_n$, such that $n' \leq r - 1$ and $|\sum\limits_{i=1}^{n'} A_{j_i}| = r$.*

The following can be thought of as a composite analog of the Cauchy-Davenport Theorem [22]. Observe that Theorem 2.6 implies $|\sum_{i=1}^{n} A'_i| \geq \min\{m,\, |S|-n+1\}$ unless $N(A', H_a) > 0$ and H_a is a proper, nontrivial subgroup.

Theorem 2.6. *Let S be a finite sequence of elements from an abelian group G with an n-set partition $A = A_1, \ldots, A_n$. Then there exists an n-set partition $A' = A'_1, \ldots, A'_n$ of S whose sumset is a union of H-cosets for some (possibly trivial) finite subgroup H of G, such that*

$$\left|\sum_{i=1}^{n} A'_i\right| \geq \big(E(A', H) + (N(A', H) - 1)n + 1\big)\,|H|,$$

where $N(A', H) = \frac{1}{|H|}|\bigcap_{i=1}^{n}(A'_i + H)|$, and where $E(A', H) = \sum_{j=1}^{n}(|A'_j| - |A'_j \cap \bigcap_{i=1}^{n}(A'_i + H)|)$.

We conclude the list of tools with a strengthening of a recent theorem of [22].

Theorem 2.7. *Let S be a sequence of elements from an abelian group G of order m with an n-set partition $P = P_1, \ldots, P_n$, and let p be the smallest prime divisor of m. [Suppose that $n' \geq \frac{m}{p} - 1$, that $|S| \geq m + \frac{m}{p} + p - 3$, and that P has at least $n - n'$ cardinality one sets.] Then either:*

(i) there exists an n-set partition $A = A_1, A_2, \ldots, A_n$ of S [with at least $n - n'$ cardinality one sets], such that:

$$|\sum_{i=1}^{n} A_i| \geq \min\{m,\, |S| - n + 1\};$$

(ii) (a) there exists $\alpha \in G$ and a nontrivial proper subgroup H_a of index a such that all but at most $\min\{a-2, \left\lfloor \frac{|S|-n}{|H_a|} \right\rfloor - 1\}$ *terms of S are from the coset $\alpha + H_a$; and (b) there exists an n-set partition $A_1, A_2, \ldots, A_n$ of the subsequence of S consisting of terms from $\alpha + H_a$ such that $\sum_{i=1}^{n} A_i = n\alpha + H_a$.*

Proof. The theorem without the bracketed parts of (i), with the quantity $\min\{a-2, \lfloor \frac{|S|-n}{|H_a|} \rfloor - 1\}$ replaced by the quantity $a-2$, and with the quantity $\min\{m,\, |S|-n+1\}$ replaced by the quantity $\min\{m,\, (n+1)p,\, |S|-n+1\}$, which we shall refer to in this paper as Theorem 2.7', was proved in [22]. We shall derive Theorem 2.7 as a consequence of Theorems 2.6 and 2.7'. We may assume

$|S'| \geq n+1$, since otherwise (i) holds trivially. Additionally, from the definition of n' it follows that there exists a subsequence S' of S of length $|S|-n+n'$ with an n'-set partition B; we will call any such subsequence a *good* subsequence. Assume the theorem is false. To simplify calculations, in the remainder of the proof we implicitly use the fact that since the number of terms e that lie outside a coset $\alpha + H_a$ is an integer, it follows that $e \leq \min\{a-2, \lfloor \frac{|S|-n}{|H_a|} \rfloor - 1\}$ if and only if $e \leq \min\{a-2, \frac{|S|-n}{|H_a|} - 1\}$.

Step 1: We show that if (ii)(a) holds for S with subgroup H_a, then (ii) holds for S with a subgroup $H_{ka} \leq H_a$. Let T be the subsequence of S consisting of all terms from the coset $\alpha + H_a$ from (ii)(a), and let T' be the sequence obtained by adding $-\alpha$ to each term of T. Since at most $a-2$ terms of S are not from $\alpha + H_a$, and since $|S| \geq n + \frac{m}{p} + p - 3 \geq n + \frac{m}{a} + a - 3$, it follows that $|T'| = |T| \geq n + \frac{m}{a} - 1$. Hence, since $n \geq n' \geq \frac{m}{p} - 1$, it follows that if (i) holds from Theorem 2.7$'$ when applied to T' with its terms considered as elements of H_a, then (ii)(b) follows for S with subgroup H_a (note that the n-set partition of S induces an n''-set partition of T' for some $n'' \leq n$, and since $|T'| \geq n$, it then follows that this n''-set partition is easily modified to become an n-set partition of T'). If instead Theorem 2.7'(ii) holds for T' with some proper nontrivial subgroup H_{ka} of index k in H_a, then there are at most

$$k - 2 + \min\{a-2, \frac{|S|-n}{|H_a|} - 1\} = \min\{k-2+a-2,\ k-2+\frac{a(|S|-n)}{m} - 1\} \leq \min\left\{ka-4,\ \frac{ka(|S|-n)}{m} - 1 + \left(k-2-(k-1)\frac{a(|S|-n)}{m}\right)\right\},$$

terms from S not from the corresponding coset of H_{ka}. If $k-2-(k-1)\frac{a(|S|-n)}{m} \geq 0$, then from $|S| \geq n + \frac{m}{p} + p - 3 \geq n + \frac{m}{a} + a - 3$, it follows that $0 \geq \frac{m}{a} + (k-1)(a-3)$, which since $k \leq \frac{m}{2}$ and since $a \geq 2$, is impossible. Therefore $k-2-(k-1)\frac{a(|S|-n)}{m} < 0$, whence it follows that there are at most $\min\{ka-4, \frac{(|S|-n)}{|H_{ka}|} - 1\}$ terms from S not from the corresponding coset of H_{ka}. Thus (ii) holds with H_{ka} for all of S, and we conclude that it suffices to show (ii)(a) holds for S to conclude (ii) holds for S.

Step 2: Next we show that (ii)(a) must hold for any good subsequence S'. Since $n \geq n' \geq \frac{m}{p} - 1$, it follows that $(n+1)p \geq (n'+1)p \geq m$, and thus the term $(n+1)p$ can be dropped from the inequality in Theorem 2.7'(i). Hence, since (i) does not hold for S, then every n'-set partition, $B = B_1, \ldots, B_{n'}$, of every good subsequence S' is such that

$$\left|\sum_{i=1}^{n'} B_i\right| < \min\{m,\ (n'+1)p,\ |S'| - n' + 1\} = \min\{m,\ |S'| - n' + 1\}.$$

Apply Theorem 2.6 with S' and B, and let $A' = A'_1, \ldots, A'_{n'}$ be the resulting n'-set partition and let H_a be the resulting subgroup of index a. In view of the comments just above Theorem 2.6 and the above inequality, it follows that may assume that H_a is a proper, nontrivial subgroup, and that $N(A', H_a) > 0$; furthermore, since $n' \geq \frac{m}{p} - 1$, implying $(n'+1)p \geq m$, we can also assume $N(A', H_a) < 2$. Hence $N(A', H_a) = 1$, implying that all but at most $E(A', H_a)$ terms of S' are from the same H_a-coset, and that $|\sum_{i=1}^{n'} A'_i| \geq (E(A', H_a) + 1)|H_a|$. Thus, since $\left|\sum_{i=1}^{n'} A'_i\right| < \min\{m,\ |S'| - n' + 1\}$, it follows that we can assume $E(A', H_a) \leq \min\{a-2, \frac{|S'|-n'}{|H_a|} - 1\}$. Hence (ii)(a) holds for S' with subgroup H_a.

Step 3: Observe that if $\alpha + H_a$ is any coset for which (ii)(a) holds for a good subsequence S' and if there does not exist a term $b \in S \setminus S'$ such that $b \notin \alpha + H_a$, then (ii)(a) follows for S, and the proof is complete in view of Step 1. So we can always assume otherwise. Hence in this situation we can remove some term b' from S' that is from $\alpha + H_a$ and include b in one of the terms of B (possible since there are at most $a - 2 \leq \frac{m}{p} - 2$ terms not from $\alpha + H_a$ in B and $n' \geq \frac{m}{p} - 1$) to obtain a new n'-set partition B' of the new good subsequence $S'' = (S' \setminus b') \cup b$.

Step 4: Let $\alpha + H_a$ be a minimal cardinality coset of index a for which there exists a good subsequence that satisfies (ii)(a) with $\alpha + H_a$. In view of Step 2 such a coset exists. Choose a good subsequence S' such that there are exactly $\min\{a-1, \lfloor \frac{|S'|-n'}{|H_a|} \rfloor\}$ terms not from the coset $\alpha + H_a$; possible by repeated application of the procedure from Step 3. In view of Step 2, let $\alpha' + H_{a'}$ be a coset of index a' for which (ii)(a) holds for S'.

Suppose that $|(\alpha + H_a) \cap (\alpha' + H_{a'})| \leq 1$. Since S' has an n'-set partition, it follows that multiplicity of ever term of S' is at most n'. Hence from the conclusions of the last two sentences we can assume that

$$|S'| \le n' + \min\{a-1, \frac{|S'|-n'}{|H_a|}\} + \min\{a'-2, \frac{|S'|-n'}{|H_{a'}|} - 1\} \le n' + \frac{|S'|-n'}{|H_a|} + \frac{|S'|-n'}{|H_{a'}|} - 1.$$

Hence by clearing denominators it follows that

$$|S'|(|H_a||H_{a'}| - |H_a| - |H_{a'}|) \le n'(|H_a||H_{a'}| - |H_a| - |H_{a'}|) - |H_a||H_{a'}|,$$

implying that $|S'| < n'$. However, since S' has an n'-set partition, it follows that $|S'| \ge n'$, contradicting the previous sentence. So we may assume that $|(\alpha + H_a) \cap (\alpha' + H_{a'})| \ge 2$. Hence w.l.o.g. we may assume that $\alpha = \alpha'$ and that $H_a \cap H_{a'} = H_c$ for some nontrivial subgroup H_c of index c. Thus $k_1 a = c$ and $k_2 a' = c$ for some divisors k_1 and k_2 of c.

Suppose that $k_1 \ne 1$ and $k_2 \ne 1$, i.e. that neither $H_a \le H_{a'}$ nor $H_{a'} \le H_a$ holds. Again it follows that at most

$$|S'| \le \min\{a-1, \frac{|S'|-n'}{|H_a|}\} + \min\{a'-2, \frac{|S'|-n'}{|H_{a'}|} - 1\},$$

terms of S' are not from the coset $\alpha + H_c$. Since $k_1 \ne 1$ and $k_2 \ne 1$, then $a - 1 + a' - 2 \le \frac{c}{k_1} + \frac{c}{k_2} - 3 < c - 2$. Also, since $k_1 \ne 1$ and $k_2 \ne 1$, and since $a' \le a$ follows from the minimality of H_a, then it follows that $a' \le (k_1 - 1)a$, implying that $\frac{|S'|-n'}{|H_a|} + \frac{|S'|-n'}{|H_{a'}|} - 1 \le \frac{|S'|-n'}{|H_c|} - 1$. Hence we see that at most

$$\min\{a-1, \frac{|S'|-n'}{|H_a|}\} + \min\{a'-2, \frac{|S'|-n'}{|H_{a'}|} - 1\} \le \min\{c-2, \frac{|S'|-n'}{|H_c|} - 1\}$$

terms of S' are not from the coset $\alpha + H_c$. Thus $(ii)(a)$ holds for S' with subgroup $H_c < H_a$, contradicting the minimality of H_a. So we may assume that either either $H_a \le H_{a'}$ or $H_{a'} \le H_a$. However, since S' has $\min\{a-1, \frac{|S'|-n'}{|H_a|}\}$ terms not from $\alpha + H_a$, since $\alpha = \alpha'$, and since (ii)(a) holds for S' with $\alpha + H_{a'}$, then it follows that $H_a \ne H_{a'}$. Hence by the minimality of H_a it follows that we cannot have $H_{a'} < H_a$, and so must instead have $H_a < H_{a'}$.

Step 5: Let $\alpha' + H_d$ be a minimal cardinality coset of index d for which (ii)(a) holds for some good subsequence with exactly $\min\{a-1, \lfloor\frac{|S'|-n'}{|H_a|}\rfloor\}$ terms not from the coset $\alpha + H_a$. From the above paragraph it follows that w.l.o.g. $\alpha' = \alpha$ and $H_a < H_d$. Now choose a good subsequence S' with exactly $\min\{a-1, \lfloor\frac{|S'|-n'}{|H_a|}\rfloor\}$ terms not from the coset $\alpha + H_a$, and exactly $\min\{d-1, \lfloor\frac{|S'|-n'}{|H_d|}\rfloor\}$ terms not from the coset $\alpha + H_d$; possible by repeated application of the procedure from Step 3 provided we can always choose b' from $(\alpha + H_d) \setminus (\alpha + H_a)$. We next show that such b' always exists, which is a straightforward inequality calculation given below.

Suppose no such b' exists. Then all $\min\{a-1, \lfloor\frac{|S'|-n'}{|H_a|}\rfloor\}$ terms outside $\alpha + H_a$ are also outside $\alpha + H_d$, implying $\min\{a-1, \lfloor\frac{|S'|-n'}{|H_a|}\rfloor\} \leq \min\{d-2, \lfloor\frac{|S'|-n'}{|H_d|}\rfloor - 1\}$. There are four cases. *Case 1:* If $a - 1 \leq d - 2$, this implies that the index of H_d is greater than the index of H_a, contradicting that $H_a < H_d$. *Case 2:* If $\lfloor\frac{|S'|-n'}{|H_a|}\rfloor \leq \lfloor\frac{|S'|-n'}{|H_d|}\rfloor - 1$, then this implies that $|H_d| \leq |H_a|$, again contradicting that $H_a < H_d$. *Case 3:* If $a \leq \lfloor\frac{|S'|-n'}{|H_a|}\rfloor$ and $d - 2 \geq \lfloor\frac{|S'|-n'}{|H_d|}\rfloor$, then it follows from the first inequality that $m \leq |S'| - n'$, while it follows from the second inequality that $m - |H_d| \geq |S'| - n'$, a contradiction to $m \leq |S'| - n'$. *Case 4:* If $a - 2 \geq \lfloor\frac{|S'|-n'}{|H_a|}\rfloor$ and $d \leq \lfloor\frac{|S'|-n'}{|H_d|}\rfloor$, then it follows from the first inequality that $m - |H_a| \geq |S'| - n'$, while it follows from the second inequality that $m \leq |S'| - n'$, a contradiction to $m - |H_a| \geq |S'| - n'$. Consequently, we see in all cases that the inequality $\min\{a-1, \lfloor\frac{|S'|-n'}{|H_a|}\rfloor\} \leq \min\{d-2, \lfloor\frac{|S'|-n'}{|H_d|}\rfloor - 1\}$ yields a contradiction. So we may assume that we can always choose b' from $(\alpha + H_d) \setminus (\alpha + H_a)$.

Step 6: Let k be the maximal integer for which there exists a chain of distinct nontrivial proper subgroups of indices a_i, $H_{a_1} < H_{a_2} < \ldots < H_{a_k}$, an element $\alpha \in G$, and a good subsequence S' with exactly $\min\{a_i - 1, \lfloor\frac{|S'|-n'}{|H_{a_i}|}\rfloor\}$ terms not from the coset $\alpha + H_{a_i}$ for each i; such that for all $j > 1$ the coset for which (ii)(a) holds with minimal cardinality over all good subsequences with exactly $\min\{a_i - 1, \lfloor\frac{|S'|-n'}{|H_{a_i}|}\rfloor\}$ terms not from the coset $\alpha + H_{a_i}$ for each $i < j$, is $\alpha + H_{a_j}$; and such that for all $j < k$, given any good subsequence S'' with exactly $\min\{a_i - 1, \lfloor\frac{|S'|-n'}{|H_{a_i}|}\rfloor\}$ terms not from the coset $\alpha + H_{a_i}$ for each $i \leq j$, and given a coset $\alpha' + H_{a'}$ for which (ii)(a) holds for S'', then $\alpha + H_{a'} = \alpha' + H_{a'}$, so that w.l.o.g $\alpha = \alpha'$, and $H_{a_j} < H_{a'}$. We have just seen in Steps 4 and 5 that these conditions are met for $k = 2$ with $H_{a_2} = H_d$ and $H_{a_1} = H_a$. Hence, since G is finite, such a maximal k exists.

In view of Step 2, let $\alpha' + H_{a'}$ be a coset of index a' for which (ii)(a) holds for a subsequence

S' that satisfies the conditions from the definition of k. From the definition of k, it follows that w.l.o.g. $\alpha' = \alpha$ and $H_{a_{k-1}} < H_{a'}$. Let $H_c \stackrel{def}{=} H_{a'} \cap H_{a_k}$. Note $H_{a_{k-1}} \leq H_c$, implying H_c is nontrivial. Since there are exactly $\min\{a_k - 1, \lfloor \frac{|S'|-n'}{|H_{a_k}|} \rfloor\}$ terms of S' not from $\alpha + H_{a_k}$, and since (ii)(a) holds for S' with $\alpha + H_{a'}$, it follows that $H_{a'} \neq H_{a_k}$. Hence by the minimality of H_{a_k} it follows that $H_{a'} \nleq H_{a_k}$. Suppose $H_{a_k} \nleq H_{a'}$. Then $H_{a_k} \nleq H_{a'}$ and $H_{a'} \nleq H_{a_k}$, and thus by calculations almost identical to those done in the third paragraph of Step 4, it follows that (ii)(a) holds with $H_c < H_{a_k}$, contradicting the minimality of H_{a_k}. So we can assume $H_{a_k} < H_{a'}$.

Choose a good subsequence with exactly $a_i - 1$ terms not from the coset $\alpha + H_{a_i}$ for each $i \leq k$, such that (ii)(a) holds for S' with minimal cardinality coset $\alpha' + H_{a_{k+1}}$ of index a_{k+1}. From the above paragraph it follows that w.l.o.g. $\alpha' = \alpha$ and $H_{a_k} < H_{a_{k+1}}$. Now choose a good subsequence with exactly $a_i - 1$ terms not from the coset $\alpha + H_{a_i}$ for each $i \leq k+1$; possible by repeated application of the procedure from Step 3 provided we can always choose b' from $(\alpha + H_{a_{k+1}}) \setminus (\alpha + H_{a_k})$. The calculation that such b' exists is almost identical to the similar calculation done in Step 5. Hence from the conclusion of the last paragraph we see that $k+1$ contradicts the maximality of k, and the proof is complete □

3 General upper and lower bounds

Theorem 3.1. *Let m, j be integers satisfying $2 \leq j \leq m$, and let $k = \left\lfloor \frac{-1+\sqrt{\frac{8m-9+j}{j-1}}}{2} \right\rfloor$. Then $f_j(m,2) \geq 4m - 2 + (j-1)k$.*

Proof. Consider the coloring $\Delta : [1, 4m-3+(j-1)k] \to \{0,1\}$ given by

$$0^{m-1-(j-1)\frac{k(k+1)}{2}}(1^{j-1}0^{k(j-1)})(1^{j-1}0^{(k-1)(j-1)})\cdots(1^{j-1}0^{2(j-1)})(1^{j-1}0^{j-1})1^{2m-1}0^{m-1} .$$

Using the quadratic formula, it can be easily verified that k is the greatest integer such that $\sum_{i=1}^{k}(j-1)i = (j-1)\frac{k(k+1)}{2} \leq m-1$. Thus,

$$\left|\Delta^{-1}(0) \cap [1, m-1+(j-1)k]\right| = m-1 \text{ and } \left|\Delta^{-1}(1) \cap [1, m-1+(j-1)k]\right| = (j-1)k \leq m-1.$$

Suppose there exist sets B_1, B_2 which are an (m,j)-solution. Notice that $\Delta(B_1) \neq 0$, since otherwise

$|[\max(B_1)+1, 4m-3+(j-1)k]| \leq m-2$. Similarly, $\Delta(B_2) \neq 0$. Thus $\Delta(B_i) = 1$ for $i = 1, 2$. Furthermore, given any m-set B with $\Delta(B) = 1$, there exists an m-set B^* with $\Delta(B^*) = 1$ satisfying $\max(B^*) \leq \max(B)$, $g_j(B^*) \leq g_j(B)$, and $(j-1)|g_j(B^*)$ (simply compress the set B inwards until the first j integers are consecutive with the exception of one gap of length $t(j-1)$ where a single block of zero's prevents further compression). Therefore we may assume $g_j(B_1) = j-1+t(j-1)$ for some $t \in \{0, 1, \ldots, k\}$. Since $\max(B_1) < \min(B_2)$, it follows that B_2 is contained within the last $2m-1+t(j-1)-m$ integers colored by 1, i.e. that

$$B_2 \subset \{\Delta^{-1}(1) \cap [\mathrm{II}\,(m-1+(j-1)t, 1)\,, 4m-3+(j-1)k]\}.$$

Hence, since $\left|\Delta^{-1}(1) \cap [1, m-1+(j-1)k]\right| = (j-1)k \leq m-1$ forces B_2 to be contained in the block of $2m-1$ consecutive integers colored by 1, it follows that

$$g_j(B_2) \leq (j-1) + (m-1+(j-1)t) - m = (t+1)(j-1) - 1.$$

Consequently, $g_j(B_1) > g_j(B_2)$, a contradiction. □

Remark: Theorem 3.1 yields the lower bounds $f_m(m,2) \geq 5m-3$ and $f_{m-1}(m,2) \geq 5m-4$. It is shown in [8] that the former lower bound is sharp, and we show in this paper that the latter lower bound is sharp for $m \geq 9$ as well. Therefore, the construction given in Theorem 3.1 is the best possible in some cases.

Lemma 3.2. *Let m, j be integers satisfying $2 \leq j \leq m$. If $\Delta : [1, 3m-2] \to \{0, 1\}$ is an arbitrary coloring, then one of the following holds:*
(i) there exists a monochromatic m-set $B \subset [1, 3m-2]$ satisfying $g_j(B) \geq m+j-2$,
(ii) there exists an (m, j)-solution,
(iii) the coloring Δ is given (up to symmetry) by 1^r0H, for some $r \in [j, m-1]$, and H is a block such that there exists a monochromatic m-set $B \subset 0H$ for which $g_j(B) \geq m+2j-r-3$.

Proof. Assume w.l.o.g. $\Delta(1) = 1$. If $\left|\Delta^{-1}(1)\right| < m$, then $\left|\Delta^{-1}(0)\right| \geq 2m-1$, whence (i) follows. So $\left|\Delta^{-1}(1)\right| \geq m$. Let $S = [m+j-1, 3m-2]$. Since $\Delta(1) = 1$ and $\left|\Delta^{-1}(1)\right| \geq m$, it follows that if $\left|\Delta^{-1}(1) \cap S\right| \geq m-j+1$, then (i) follows. Hence $\left|\Delta^{-1}(1) \cap S\right| \leq m-j$, whence

$$\left|\Delta^{-1}(0) \cap S\right| \geq m. \tag{1}$$

Let $y_2 < y_3 < \cdots < y_m \in \{\Delta^{-1}(0) \cap S\}$ be a final segment. Observe, since $|\Delta^{-1}(1) \cap S| \leq m - j$, that $y_j \geq m + 2j - 2$. Hence, if there exists $i \in [1, j]$ such that $\Delta(i) = 0$, then (i) follows. Consequently, $\Delta(i) = 1$ for $i \in [1, j]$. However, if $\Delta(i) = 1$ for $i \in [1, m]$, then (ii) follows in view of (1). Therefore, there exists a minimal $i \in [j + 1, m]$ such that $\Delta(i) = 0$. Define $r = i - 1$. Then the set $B = \{r + 1, y_2, \ldots, y_m\}$ satisfies $g_j(B) \geq m + 2j - 2 - (r + 1) = m + 2j - r - 3$, whence (iii) follows. □

Theorem 3.3. *Let m, j be integers satisfying $2 \leq j \leq m$. Then $f_j(m, 2) \leq 5m - 3$.*

Proof. Let $\Delta : [1, 5m - 3] \to \{0, 1\}$ be an arbitrary coloring. Apply Lemma 3.2 to the interval $[2m, 5m - 3]$. If Lemma 3.2(ii) holds, then the proof is complete, and if Lemma 3.2(i) holds, then by applying the pigeonhole principle to $[1, 2m - 1]$ the proof is also complete. Thus we may assume Lemma 3.2(iii) holds, so that w.l.o.g.

$$\Delta([2m, 5m - 3]) = 1^r 0H,$$

where r and H are as in Lemma 3.2(iii), and that there is a subset $B \subset [2m + r, 5m - 3]$ with $g_j(B) \geq m + 2j - r - 3$. Let $S = [1, 2j - 1]$.

Case 1: $|\Delta^{-1}(1) \cap S| \geq j$.

Since $r \leq m - 1$, it follows that $g_j(B) \geq 2j - 2$. Hence we may assume

$$|\Delta^{-1}(1) \cap [1, 2m + r - 1]| \leq m - 1.$$

But then since $\Delta([2m,\, 2m + r - 1]) = 1$, it follows that

$$|\Delta^{-1}(1) \cap [2j, 2m - 1]| \leq m - j - r - 1, \tag{2}$$

implying, since $j \leq r$, that

$$|\Delta^{-1}(0) \cap [2j, 2m - 1]| \geq m - j + r + 1 \geq m.$$

Let $y_1, y_2, \ldots, y_m \in \{\Delta^{-1}(0) \cap [2j, 2m - 1]\}$ be an initial segment. Then by (2), it follows that $B_1 = \{y_1, \ldots, y_m\}$ is a monochromatic m-set with $g_j(B_1) \leq m - r - 2$, whence B_1, B are an

(m, j)-solution.

Case 2: $\left|\Delta^{-1}(0) \cap [1, 2j-1]\right| \geq j$.

It follows, as in Case 1, that

$$\left|\Delta^{-1}(0) \cap [1, 2m+r-1]\right| \leq m-1. \tag{3}$$

Let d be the positive integer such that r is contained in the interval

$$\frac{(d-1)m + dj - d + 1}{d} \leq r < \frac{dm + (d+1)j - (d+1) + 1}{d+1}; \tag{4}$$

note, since

$$\lim_{d \to \infty} \frac{(d-1)m + dj - d + 1}{d} = m + j - 1 > m,$$

and since in view of Lemma 3.2(iii) we have $j \leq r < m$, it follows that such a d exists. Also note that if $j \geq \frac{m}{d}$, then (4) implies $m - 1 < r$, a contradiction. Hence we may assume $j < \frac{m}{d}$. From (3) and (4), it follows that

$$\left|\Delta^{-1}(1) \cap [1, 2m+r-1]\right| \geq m + r \geq m + \frac{(d-1)m + dj - d + 1}{d}. \tag{5}$$

But then, letting b be the $m - j + 1$ greatest integer colored by 1 in $[1, 2m+r-1]$, since $j < \frac{m}{d}$, it follows from (5) that

$$\left|\Delta^{-1}(1) \cap [1, b]\right| \geq \frac{(d-1)m + dj - d + 1}{d} + j \geq (d+1)(j-1) + 1.$$

Hence let $z_1 < z_2 < \cdots < z_{m-j} \in \{\Delta^{-1}(1) \cap [1, 2m+r-1]\}$ be a final segment, and let $y_1 < y_2 < \cdots < y_{(d+1)(j-1)+1} \in \{\Delta^{-1}(1) \cap [1, 2m+r-1]\}$ be an initial segment. If for some index $i \in [0, d]$

$$\left|\Delta^{-1}(0) \cap [y_{i(j-1)+1}, y_{(i+1)(j-1)+1}]\right| \leq m + j - r - 2,$$

then $B_1 = \{y_{i(j-1)+1}, y_{i(j-1)+2}, \ldots, y_{(i+1)(j-1)+1}, z_1, z_2, \ldots, z_{m-j}\}$ is a monochromatic m-set with $g_j(B_1) \leq m + 2j - r - 3 = g_j(B)$, whence B_1, B are an (m, j)-solution, and the proof is complete.

Therefore, we may assume that

$$\left|\Delta^{-1}(0) \cap [y_{i(j-1)+1}, y_{(i+1)(j-1)+1}]\right| \geq m + j - r - 1 \quad \text{for } i = 0, 1, \ldots, d.$$

But then the above inequalities and (4) imply that

$$\left|\Delta^{-1}(0) \cap [1, 2m-1]\right| \geq (d+1)(m+j-r-1) > m-1,$$

contradicting (3), and completing the proof. □

Corollary 3.4. *$F(m, 2) = 5m - 3$.*

Proof. Theorem 3.1 with $j = m$ implies that $f_m(m, 2) \geq 5m - 3$, whence $F(m, 2) \geq 5m - 3$. Theorem 3.3 implies that $F(m, 2) \leq 5m - 3$, as needed. □

Lemma 3.5. *Let m, j be integers satisfying $2 \leq j \leq m$, and let $\Delta : [1, 4m-3] \to \mathbb{Z}_m$ be an arbitrary coloring.*
(i) If m is prime, then there exists a zero-sum m-set $B \subset [1, 4m-3]$ with $g_j(B) \geq m + j - 2$;
(ii) If $j \geq \frac{m}{p} + p - 1$, where p is the smallest prime divisor of m, then there exists a zero-sum m-set $B \subset [1, 4m-3]$ with $g_j(B) \geq m + j - 2$.

Proof. Consider the interval $S = [m+1, 4m-3]$. If there does not exist a $(2m-2)$-set partition of the sequence $\Delta(S)$ with $m-1$ sets of cardinality 2, then since $|S| = 3m-3$, it follows that there exists $a \in \mathbb{Z}_m$ such that

$$\left|\Delta^{-1}(a) \cap S\right| \geq 2m - 1 \quad \text{and} \quad \left|\Delta^{-1}(\mathbb{Z}_m \setminus \{a\}) \cap S\right| \leq m - 2.$$

Let $y_1 < y_2 < \cdots < y_{2m-1} \in \Delta^{-1}(a) \cap S$. Define $B = \{y_1, \ldots, y_{j-1}, y_{m+j-1}, y_{m+j}, \ldots, y_{2m-1}\}$. Then $g_j(B) \geq m + j - 2$, and the proof is complete. So we may assume that there exists a $(2m-2)$-set partition P of the sequence $\Delta(S)$ with $(m-1)$ sets of cardinality 2.

Suppose first that m is prime. Define $x_1 = 1$. Applying the Cauchy-Davenport theorem to P, it follows that there exist integers $x_2 < x_3 < \cdots < x_m \in S$ such that $\sum_{i=2}^{m} \Delta(x_i) = -\Delta(x_1)$. Thus, $(x_1, \ldots, x_m)$ is zero-sum. Furthermore, by definition of the x_i's, we have $x_j \geq m + 1 + (j-2) = m + j - 1$, so that $B = \{x_1, \ldots, x_m\}$ satisfies $g_j(B) \geq m + j - 2$, and (i) follows.

To prove (ii), suppose $j \geq \frac{m}{p} + p - 1$, where p is the smallest prime divisor of m. Applying Theorem 2.7 to P, it follows that either Theorem 2.7(i) holds and there exist integers $x_2, \ldots, x_m \in S$ such that $(1, x_2, x_3 \ldots, x_m)$ is zero-sum (note the resulting $(2m-2)$-set partition from Theorem 2.7(i) will have at most $m-1$ set with cardinality greater than one; hence since by Theorem 2.7(i) we have that the cardinality of the sumset of that $(2m-2)$-set partition is at least m, then given any one of the m elements from $\mathbb{Z}_m$ it follows that we can find a selection of $m-1$ terms from the resulting set partition, including one from each set with cardinality greater than one, which sum to the additive inverse of that element), whence the proof is complete as above; or else Theorem 2.7(ii) holds and there exists a coset, which w.l.o.g. we may assume by translation is a subgroup, say $a\mathbb{Z}_m = H_a$, such that all but at most $a-2$ terms of the sequence $\Delta(S)$ are elements of H_a, whence it follows from Theorem 2.3 that any subset $T \subset S$ satisfying $|T| \geq m + \frac{m}{a} - 1 + (a-2)$ contains a zero-sum m-tuple. Let

$$S_1 = [m+1, m + \frac{m}{p} + p - 2] \quad \text{and} \quad S_2 = [3m-1, 4m-3].$$

Since $|S_1 \cup S_2| = m + \frac{m}{p} + p - 3 \geq m + \frac{m}{a} - 1 + (a-2)$, it follows that there exist m integers $x_1 < x_2 < \cdots < x_m \in S_1 \cup S_2$ such that $\sum_{i=1}^{m} \Delta(x_i) = 0$. Since $|S_2| = m-1$, we must have $x_1 \in S_1$. Furthermore, since $|S_1| = \frac{m}{p} + p - 2 \leq j - 1$, we must have $x_j \in S_2$. Hence it follows that $B = \{x_1, \ldots, x_m\}$ is a zero-sum m-set satisfying $g_j(B) \geq m + j - 2$, whence (ii) is satisfied. □

Lemma 3.6. *Let m, j be positive integers satisfying $2 \leq j \leq m$, let p be the smallest prime divisor of m, and let $\Delta : [1, 6m + \frac{m}{p} - 5] \to \mathbb{Z}_m$ be an arbitrary coloring. Then one of the following holds:*
(i) there exists a zero-sum m-set $B \subset [1, 6m + \frac{m}{p} - 5]$ satisfying $g_j(B) \geq m + j - 2$;
(ii) there exists an $(m, j, \mathbb{Z}_m)$-solution.

Proof. Let D denote the sequence $\left(\Delta\left(m + \frac{m}{p}\right), \Delta\left(m + \frac{m}{p} + 1\right), \ldots, \Delta\left(4m + \frac{m}{p} - 4\right)\right)$. In view of the arguments from the third paragraph of the proof of Lemma 3.5, applied to the interval $[m + \frac{m}{p}, 4m + \frac{m}{p} - 4]$ rather than $[m+1, 4m-3]$, we may assume that there exists a subgroup, say $H_a = a\mathbb{Z}_m$, such that all but at most $a-2$ terms of D are all elements of H_a, and, furthermore, that there exists a $(2m-2)$-set partition P_1 of the terms of D which are elements of H_a such that the sumset of P_1 is H_a. Finally, from Theorem 2.1 it follows that from among the sequence

$$(\Delta(1), \Delta(2), \Delta(3), \cdots, \Delta(a))$$

we can find a subsequence D_1 of length $1 \leq q \leq a$ whose terms are consecutive and whose sum is an element $h \in H_a$.

Case 1: $q < j$.

From Proposition 2.5 it follows, by selectively deleting terms from P_1, that we can find an $(m-q)$-set partition P_2 of a subsequence D_2 of D such that the sumset of P_2 is still H_a. Consequently, we can find an $m-q$ terms of D_2 with sum $-h$, which, together with the terms of D_1, gives an m-element zero-sum subset B with $g_j(B) \geq m+j-2$.

Case 2: $q \geq j$.

By the arguments in Case 1, we can find an m-element zero-sum set $B_1 \subset [1, 4m+\frac{m}{p}-4]$ which includes all $q \geq j$ consecutive elements of D_1, and hence $g_j(B_1) \leq j-1$. By Theorem 0, there exists an m-element zero-sum set $B_2 \subset [4m+\frac{m}{p}-3, 6m+\frac{m}{p}-5]$. Since B_1, B_2 are an $(m, j, \mathbb{Z}_m)$-solution, the proof is complete. □

Theorem 3.7. *Let m, j be integers satisfying $2 \leq j \leq m$.*
(i) If m is prime, then $f_j(m, \mathbb{Z}_m) \leq 6m-4$.
(ii) If $j \geq \frac{m}{p}+p-1$, where p is the smallest prime divisor of m, then $f_j(m, \mathbb{Z}_m) \leq 6m-4$.
(iii) $f_j(m, \mathbb{Z}_m) \leq 8m+\frac{m}{p}-6$.

Proof. Let $s \in \{6m-4, 8m+\frac{m}{p}-6\}$, and let $\Delta : [1, s] \to \mathbb{Z}_m$ be an arbitrary coloring. By Theorem 0, there exists a zero-sum m-set $B \subset [1, 2m-1]$, and, furthermore, we have $g_j(B) \leq m+j-2$. The proof of (i) and (ii) is complete by letting $s = 6m-4$ and applying Lemma 3.5 (i) or (ii) to $[2m, s]$, respectively. To show (iii), set $s = 8m+\frac{m}{p}-6$, and apply Lemma 3.6 to $[2m, s]$. □

Corollary 3.8. $\liminf\limits_{m\to\infty} \frac{F(m,\mathbb{Z}_m)}{F(m,2)} \leq 1.2$.

Proof. The result follows from Corollary 3.4 and Theorem 3.7(i). □

4 Determination of $f_{m-1}(m,2)$ and $f_{m-1}(m,\mathbb{Z}_m)$

For notational convenience, let $f(m,2)$ and $f(m,\mathbb{Z}_m)$ denote $f_{m-1}(m,2)$ and $f_{m-1}(m,\mathbb{Z}_m)$, respectively. Furthermore, let g denote the function g_{m-1}. Finally, we use the terminology m-solution and $(m,\mathbb{Z}_m)$-solution for $(m,m-1)$-solution and $(m,m-1,\mathbb{Z}_m)$-solution, respectively.

Lemma 4.1. *Let $m \geq 3$ be an integer, and let $\Delta : [1,3m-3] \to \{0,1\}$ be a coloring. Then one of the following holds:*
(i) there exists a monochromatic m-set $B \subset [1,3m-3]$ with $g(B) \geq 2m-4$;
(ii) there exists an m-solution;
(iii) the coloring Δ is given (up to symmetry) by $1^{m-1}0^{2m-3}1$ or $1^{m-1}0^{2m-4}10$.

Proof. The proof is analogous to that of Lemma 3.2 with $j = m-1$, and we omit it. □

Lemma 4.2. *Let $m \geq 9$ be an integer, and let $\Delta : [1,3m-3] \to \mathbb{Z}_m$ be a coloring. Then one of the following holds:*
(i) there exists a zero-sum m-set $B \subset [1,3m-3]$ with $g(B) \geq 2m-3$;
(ii) there exists an $(m,\mathbb{Z}_m)$-solution;
(iii) Δ is given up to affine transformation by $1^{m-2}21^{m-2}0^m$, $1^{m-1}21^{m-3}0^m$ or $1^{m-3}21^{m-1}0^m$;
(iv) Δ is given up to affine transformation by $1^{m-1}0H$, where H is a block such that there exists $B \subset 0H$ satisfying $g(B) = 2m-4$;
(v) Δ reduces to monochromatic.

Proof. Let m, Δ be as above. Define $S_1 = \{1, 3m-4, 3m-3\}$ and observe that if there exists a zero-sum m-set which uses all the elements of S_1, then (i) follows. Let $S = [1,3m-3] \setminus S_1$, and let D be the sequence $(\Delta(2), \Delta(3), \ldots, \Delta(3m-5))$.

Case 1: $\Delta([1,3m-3]) = \{0,1,2\}$ and $\left|\Delta^{-1}(2)\right| = 1$.

Note that $|\Delta^{-1}(1)| \geq m-2$, as otherwise (v) follows. Therefore there is a zero-sum m-set B satisfying $|B \cap \Delta^{-1}(2)| = 1$, $|B \cap \Delta^{-1}(1)| = m-2$, and $|B \cap \Delta^{-1}(0)| = 1$ that contains $\{1,a,b\}$ for some distinct $a,b \in [2m-2,3m-3]$, and hence $g(B) \geq 2m-3$ yielding (i), unless every such triple $\{1,a,b\}$ has two of its elements colored by zero. However, this implies either that there exists a monochromatic m-set B with $1 \in B$ and $|B \cap [2m-2,3m-3]| = m-1$ yielding (i) (if $\Delta(1) = 0$), or that $\Delta([2m-2,3m-3]) = 0^m$, whence $\Delta(1) \in \{1,2\}$. Suppose $\left|\Delta^{-1}(0)\right| = m$. Then it is easy to see that (iii) holds unless there are m consecutive 1's, in which case (ii) follows. Therefore, we may assume that $\left|\Delta^{-1}(0)\right| \geq m+1$. Then $0 \notin \Delta([1, m-1])$ as otherwise (i) follows. Thus $2 \notin \Delta([1, m])$ as otherwise (ii) follows (take for your first set $m-1$ consecutive integers from $[1, m]$ that include an integer colored by 2 along with $\Pi(1,0)$, and for your second set choose any other m integers colored by 0). Hence $\Delta(i)=1$ for $i \in [1, m-1]$ and $\Delta(m)=0$, whence (iv) follows with $B=\{m\} \cup [2m-1, 3m-3]$.

Case 2: There does not exist $Q \subseteq [1, 3m-3]$ with $|Q|=m+1$ and $|\Delta([1, 3m-3] \setminus Q)|=1$.

From the assumption of the case it follows that if there does not exist $x \in S$ such that $|\Delta(S \setminus x)|=2$, then we can find a $(2m-5)$-set partition P of the terms of D which has $(m-2)$ sets of cardinality 1, and consequently at most $m-3$ sets with cardinality greater than one. Then applying Theorem 2.7 to P we conclude either Theorem 2.7(i) holds and there is a zero-sum subset which includes S_1 (we need the strengthening of Theorem 2.7 here to conclude that the resulting $(2m-5)$-set partition A from Theorem 2.7(i) has at most $m-3$ sets with cardinality greater than one, thus allowing us to need select at most $m-3$ terms from the set partition A in order to find our selection of terms from A whose sum will be the inverse of the sum of the terms from S_1), yielding (i), or that Theorem 2.7(ii) holds and all but at most $a-2+3$ of the elements of $[1, 3m-3]$ are colored by elements from the same coset $(a\mathbb{Z}_m+\alpha)$ of $\mathbb{Z}_m$. But in the latter case, Theorem 2.3 implies that any subset of $[1, 3m-3]$ of cardinality $(m+\frac{m}{a}-1+a+1)$ must contain a zero-sum m-set. Hence there is a zero-sum m-set

$$B \subseteq [1, m-2] \cup [3m-4-a-\frac{m}{a},\ 3m-3],$$

and as $\frac{m}{a}+a+2 \leq m-1$ for $m \geq 9$, it follows that

$$g(B) \geq 3m-5-a-\frac{m}{a} \geq 2m-2,$$

whence (i) follows.

So we may assume that there exists $x \in S$ such that $|\Delta(S \setminus x)|=2$. Now, one of the sets $S_2=\{2, 3m-5, 3m-6\}$, $S_3=\{3, 3m-5, 3m-6\}$, $S_4=\{2, 3m-7, 3m-6\}$ or $S_5=\{2, 3m-7, 3m-5\}$, say S_3, does not contain x. We may apply the arguments of the preceding paragraph to $S'=[1, 3m-3] \setminus S_3$ and conclude that $[1, 3m-3] \setminus \{x\}$ must be colored by two residue classes, say α_1, α_2, since otherwise (i) or (v) follow. Furthermore, we conclude that $\Delta(x) = \beta \notin \{\alpha_1, \alpha_2\}$ as otherwise (v) again follows.

Let $\alpha_1 - \alpha_2 = a$. If $\gcd(a, m) \neq 1$, then Theorem 2.3 implies that any subset of $[1, 3m-3]$ of cardinality $m + \frac{m}{a} - 1 + 1$ contains a zero-sum m-set, whence the proof is complete by the arguments at the end of the first paragraph of Case 2. So, $\gcd(a, m) = 1$, and hence by an affine transformation we may assume that $\{\alpha_1, \alpha_2\} = \{0, 1\}$. Furthermore, if $\Delta(x)$ is not equal to 2 or -1, then there will be a zero-sum m-set B satisfying $|B \cap \{x\}| = 1$, $|B \cap \Delta^{-1}(1)| = m - \overline{\Delta(x)} \geq 2$, and $|B \cap \Delta^{-1}(0)| = \overline{\Delta(x)} - 1 \geq 2$ that contains $\{1, a, b\}$ for some distinct $a, b \in [2m-1, 3m-3]$, and hence $g_j(B) \geq 2m-2$, unless every pair $\{a,\, b\}$ satisfies $\Delta(1) = \Delta(a) = \Delta(b)$, in which case $B = \{1\} \cup [2m-1, 3m-3]$ is a monochromatic m-set B with $g_j(B) \geq 2m-2$. In both cases (i) follows. Hence, by the affine transformation exchanging 0 and 1 if $\Delta(x) = -1$, this reduces to Case 1.

Case 3: There exists $Q \subseteq [1, 3m-3]$ such that $|Q| = m+1$ and $|\Delta([1, 3m-3] \setminus Q)| = 1$.

Assume w.l.o.g. $\Delta([1, 3m-3] \setminus Q) = \{0\}$. Define $C = Q \setminus \{\Delta^{-1}(0)\}$. Observe that if $|C| \leq m-1$, then (v) follows.

First assume that $|C| = m$. Let R denote a sequence of $(m-1)$ 0's. Let S_1 range over all possible subsequences of $\Delta(C)$ of length $m-2$. Applying Theorem 2.4 to each $S_1 \cup R$, we conclude, since $|\Delta(C)| \geq 2$ else (v) follows, that there exists a zero-sum subset $C' \subset C$ such that $1 < |C'| \leq m-2$, unless w.l.o.g. $\Delta(C) = \{1, 2\}$ and $\left|\Delta^{-1}(2) \cap C\right| = 1$, which reduces to Case 1. So we may assume that $2 \leq |C'| \leq m-2$.

Let $y_1 = \Pi(1, 0)$, $y_2 = \text{II}(2, 0)$, and $y_3 = \text{II}(1, 0)$. Notice that there will be a monochromatic m-set B with $g(B) \geq 2m-3$ unless at least $m-1$ elements of C lie in $[1, y_1 - 1] \cup [y_2 + 1, 3m-3]$. Hence, since $2 \leq |C'| \leq m-2$, it follows that C' in addition to $m - |C'|$ elements colored by zero, including y_1, y_2 and y_3 (if $|C'| < m-2$) or y_1 and y_3 (if $|C'| = m-2$, $\max(C') > y_2$), or y_2 and y_3 (if $|C'| = m-2$, $\max(C') < y_2$) will form a zero-sum m-set B satisfying $g(B) \geq 2m-3$, yielding (i).

So assume that $|C| = m+1$. As above, we may assume that there exists a zero-sum subset $C' \subset C$ such that $2 \leq |C'| \leq m-2$. Since $m-2 \geq 4$, it follows that we can find C' with $|C'| \geq 3$ unless every such C' must contain the same term $z \in C$. In the former case, we are done as in the preceding paragraph. However, in the later case, applying the arguments of the second paragraph of Case 3 to $C \setminus \{z\}$, we contract the uniqueness of $z \in C'$, or we conclude w.l.o.g. that $\Delta(C \setminus \{z\}) \subseteq \{1,2\}$ and $\left|\Delta^{-1}(2) \cap (C \setminus \{z\})\right| \leq 1$. In the latter case, since z is one element of a two element zero-sum set, it follows that we must have $\Delta(z) = -1$ or $\Delta(z) = -2$. If $\Delta(z) = -2$, then we can find C' with $\Delta(C') = -21^2$, and this reduces to the case $|C'| \geq 3$. If $\Delta(z) = -1$, then there exists $x \in C$, such that every pair $\{z, c\}$ is zero-sum, $c \in C \setminus \{z, x\}$. Let $z_1, z_2 \in C$ be an initial segment, and let $z_3, z_4 \in C$ be a final segment. Since $m-1 \geq 7$, it follows that one of $[1, y_1 - 1]$ and $[y_2+1, 3m-3]$ must contain at least 4 elements of C. Hence we can choose C' so that it contains z and one of z_1 and z_2 (if $|C \cap [1, y_1 - 1]| \geq 4$) or z and one of z_3 and z_4 (if $|C \cap [y_2 + 1, 3m-3]| \geq 4$), and it follows that the proof is complete as it was in the preceding paragraph. $\square$

Lemma 4.3. *Let $m \geq 5$ be an integer and let $\Delta : [1, 5m-4] \to \mathbb{Z}_m$ be an arbitrary coloring. If there exists an integer $\gamma \geq 2m$ such that $\Delta([\gamma, \gamma+m-4]) = \{z\}$, a zero-sum m-set $B_2 \subset [\gamma, 5m-4]$ with $g(B_2) \geq 2m-4$, a zero-sum m-set $B_3 \subset [\gamma+1, 5m-4]$ with $g(B_3) \geq 2m-5$, a zero-sum m-set $B_4 \subset [\gamma + \lfloor \frac{m}{2} \rfloor, 5m-4]$ with $g(B_4) \geq m + \lceil \frac{m}{2} \rceil - 4$, an integer $r \geq \gamma + m - 3$ such that $\Delta(r) = 0$, and a zero-sum m-set $B_5 \subset [r+1, 5m-4]$, then there exists an $(m, \mathbb{Z}_m)$-solution.*

Proof. We may w.l.o.g. assume $z = 0$. Let $S = [\gamma - 2m + 1, \gamma - 1]$, $S_1 = [\gamma - 2m + 2, \gamma - 1]$ and $S_2 = [\gamma - 2m + 1, \gamma - 3] \cup \{\gamma - 1\}$. Since $g(B_2) \geq 2m - 4$, we can assume that neither S_1 nor S_2 contains a zero-sum m-set, whence Theorem 2.4(a) implies that $|\Delta(S)| = 2$. Let $S_3 = [\gamma - 2m + 4, \gamma]$. Since $g(B_3) \geq 2m - 5$, we conclude that there does not exist an zero-sum m-set in S_3, whence, since $|\Delta(S)| = 2$, in view of Theorem 2.4 it follows w.l.o.g. that $\Delta(S) = \{1, 2\}$ or $\Delta(S) = \{0, b\}$.

Suppose first that $\Delta(S) = \{1, 2\}$. Let δ be the maximal integer such that

$$s = \sum_{i=\delta}^{\gamma-1} \overline{\Delta(i)} \geq m.$$

Then $\gamma - m \leq \delta \leq \gamma - \lceil \frac{m}{2} \rceil$. Notice that $s \in \{m, m+1\}$. Furthermore, if $s = m$, then $B_1 = \{\delta, \delta+1, \ldots, \delta+m-1\}$ satisfies $g(B_1) = m-2$, whence B_1, B_4 are an $(m, \mathbb{Z}_m)$-solution.

Therefore we may assume that $s = m+1$. Suppose there exists $j \in [\delta, \gamma-1]$ such that $\Delta(j) = 1$. If m is even, then $\delta < \gamma - \lceil \frac{m}{2} \rceil$. On the other hand, if m is odd, then since $s = m+1$, it follows that there are at least two integers colored by 1 in $[\delta, \gamma - 1]$, whence $\delta < \gamma - \lceil \frac{m}{2} \rceil$ as well. Thus $B_1 = \{\delta, \delta+1, \ldots, \delta+m\} \setminus \{j\}$ is a zero-sum m-set satisfying $g(B_1) = m-1$, which together with B_4 yields an $(m, \mathbb{Z}_m)$-solution.

So we may assume that $\Delta(j) = 2$ for $j \in [\delta, \gamma - 1]$, whence m is odd as $s = m + 1$. Now, we may assume that there exists a maximal integer $\gamma - m \le \beta \le \gamma - 1$ such that $\Delta(\beta) = 1$, since otherwise $B_1 = \{\gamma - m, \gamma - m + 1, \ldots, \gamma - 1\}$ is a zero-sum m-set satisfying $g(B_1) \le m - 2$, and the proof is complete as in the preceding paragraph. If $\beta \ge \gamma - m + 1$, then there exists a zero-sum m-set $B \subset [\beta, \gamma - 1 + \frac{m-1}{2}]$ satisfying $g(B) \le \frac{3m-7}{2}$. But then B, B_4 are an $(m, \mathbb{Z}_m)$-solution. Therefore, we may assume that $\beta = \gamma - m$, whence $\Delta([\gamma - m + 1, \gamma - 1]) = 2^{m-1}$. Hence, since $B_2 \subset [\gamma, 5m - 4]$ is such that $g(B_2) \ge 2m - 4$, it follows that $\Delta(j) = 1$ for $j \in [\gamma - 2m + 3, \gamma - m]$. But then $B_1 = [\gamma - 2m + 3, \gamma - m + 1] \cup \{\gamma\}$ satisfies $g(B_1) = m - 2$, whence B_1 and B_4 form an $(m, \mathbb{Z}_m)$-solution.

So we may assume that $\Delta(S) = \{0, b\}$. By Theorem 0, there exists a zero-sum m-set $B \subset [\gamma - 2m + 1, \gamma - 1]$. Since $g(B_2) \ge 2m - 4$, we may assume that $g(B) = 2m - 3$, whence $\Delta(\gamma - 2m + 1) = \Delta(\gamma - 2) = \Delta(\gamma - 1)$ and $|\Delta^{-1}(\Delta(\gamma - 1))| = m$. If $\Delta(\gamma - 1) = 0$, then $B_1 = \{\gamma - 2, \gamma - 1, \ldots, \gamma + m - 4, r\}$ and B_5 are an $(m, \mathbb{Z}_m)$-solution. So we may assume that $\Delta(\gamma - 1) = b$. Let $y_1 < y_2 < \cdots < y_{m-1} \in \{\Delta^{-1}(0) \cap [\gamma - 2m + 1, \gamma - 2]\}$. Then $B_1 = \{y_1, y_2, \ldots, y_{m-1}, \gamma\}$ and B_3 are an $(m, \mathbb{Z}_m)$-solution. □

Theorem 4.4. *If $m \ge 9$ is an integer, then $f(m, \mathbb{Z}_m) = f(m, 2) = 5m - 4$.*

Proof. Since $f(m, 2) \le f(m, \mathbb{Z}_m)$ holds trivially, in view of Theorem 3.1 it suffices to show $f(m, \mathbb{Z}_m) \le 5m - 4$. Let $\Delta : [1, 5m - 4] \to \mathbb{Z}_m$ be an arbitrary coloring. By Theorem 0, there exists a zero-sum m-set $B \subset [1, 2m - 1]$ with $g(B) \le 2m - 3$. Therefore, applying Lemma 4.2 to $S = [2m, 5m - 4]$, we may assume that neither (i) nor (ii) hold. If (iii) holds, then the proof is complete by Lemma 4.3 with $\gamma = 2m$. If (iv) holds, then the proof is again complete by Lemma 4.3 with $\gamma = 2m$ (let $B = B_i$ for all $i \in [1, 5]$). Thus, we may assume that conclusion (v) of Lemma 4.2 holds when applied to S. Let $\Delta^* : S \to \{0, 1\}$ be the natural induced coloring whose monochromatic m-sets are all zero-sum under Δ.

Then we may apply Lemma 4.1 to S and Δ^* and assume that conclusion (ii) does not hold. Suppose first that conclusion (iii) of Lemma 4.1 holds. Then

$$\Delta^*(S) = 0^{m-1}1^{2m-4}01 \qquad \text{or} \qquad \Delta^*(S) = 0^{m-1}1^{2m-3}0,$$

implying w.l.o.g., since each color class is used at least m times, that

$$\Delta(S) = 0^{m-1}a^{2m-4}0a \qquad \text{or} \qquad \Delta(S) = 0^{m-1}a^{2m-3}0, \tag{6}$$

where $a \in \mathbb{Z}_m$. From (6) the proof is complete by Theorem 2.4 applied to $[m+2, 3m-2]$ unless $\Delta([m+2, 2m-1]) = b$, where $b \neq 0$, or w.l.o.g. $\Delta([m+2, 2m-1] \setminus \{x\}) = 1$, $\Delta(x) = 2$, for some $x \in [m+2, 2m-1]$. In the latter case, it can be checked that there is an m-element zero-sum subset $B' \subset [m+2, 3m-1]$ with $3m-1 \in B'$, and $g(B') \leq m-1$. Likewise, in the former case if $b \neq a$, then it can be checked that there is an m-element zero-sum subset $B' \subset [m+2, 3m-1]$ with $g(B') \leq m-2$. Then, since $2m-6 \geq m-1$, it follows from (6) that the proof is complete. So $\Delta([m+2, 2m-1]) = a$.

If $[5, m+1] \cap \Delta^{-1}(a) \neq \emptyset$, then there will be an m-element monochromatic in a subset $B' \subset [5, 3m-1]$, with $g(B') \leq 2m-6$, and from (6) the proof will be complete. Hence, in view of Theorem 2.4(b) applied to $[5, 2m+1]$, it follows that $\Delta([5, m+1]) = 0$, or else there exists an m-element zero-sum subset $B' \subseteq [5, 2m+1]$ with $g(B') \leq 2m-5$, whence from (6) the proof is complete. So $\Delta([5, m+1]) = 0$. Likewise, if $\Delta([1, 4]) \nsubseteq \{0, a\}$, then the proof will be complete by applying Theorem 2.4(b) to both $[1, 2m-4] \cup \{2m\}$ and $[1, 2m-3]$. So we can conclude $\Delta([1, 2m-1]) \subseteq \{0, a\}$.

If there exist integers $j_1 < j_2 \in [1, 4]$ such that $\Delta(j_i) = 0$ for $i = 1$ and $i = 2$, then $B_1 = \{j_1, j_2, 5, 6, 7, \ldots, m+1, 2m\}$ is a monochromatic m-set with $g(B_1) \leq m$, and once more the proof is complete from (6). Therefore, we can assume that there exist integers $j_1 < j_2 < j_3 \in [1, 4]$ such that $\Delta(j_i) = a$ for $i = 1, 2, 3$, whence $B_1 = \{j_1, j_2, j_3, m+2, m+3, \ldots, 2m-2\}$ is a monochromatic m-set with $g(B_1) \leq 2m-4$. However, since $\Delta(2m-1) = a$, it follows from (6) that there exists a monochromatic m-set $B_2 \subset \{2m-1\} \cup [4m-3, 5m-4]$ such that $g(B_2) \geq 2m-4$, and the proof is complete.

So we may assume that conclusion (i) of Lemma 4.1 holds. We consider two cases.

Case 1: There exists $c \in \{0, 1\}$ such that $\left|\Delta^{*-1}(c)\right| \leq m-1$.

Without loss of generality $c = 1$. It follows that $\left|\Delta^{*-1}(0)\right| \geq 2m-2$. Furthermore, we may assume that the first $2m-3$ of the integers colored by 0 are consecutive, since otherwise under Δ we obtain a zero-sum m-set B_2 satisfying $g(B_2) \geq 2m-3$, which together with B completes the proof. Applying Lemma 4.4 with $\gamma = \min\{\Delta^{-1}(b) \cap S\}$, where b is the color such that $\Delta^{-1}(b) \geq 2m-2$, completes Case 1.

Case 2: There does not exist $c \in \{0,1\}$ such that $\left|\Delta^{*-1}(c)\right| \leq m-1$.

In this case $|\Delta(S)| \leq 2$ and w.l.o.g. we may assume $\Delta(S) = \{0, a\}$ and that there exist two integers $i_1, i_2 \in [5m-6, 5m-4]$ such that $\Delta(i_1) = \Delta(i_2) = a$. Hence $x = \min\{\Delta^{-1}(a) \cap S\}$ satisfies $x \geq 3m-2$, as otherwise there will be an m-set B_2 monochromatic in a satisfying $g(B_2) \geq 2m-3$, which along with B completes the proof. Notice that $x \leq 3m-1$ as otherwise $[2m, 3m-1]$ is a monochromatic m-set which along with any m elements colored by a form an $(m, \mathbb{Z}_m)$-solution. But then since conclusion (i) of Lemma 4.1 holds for $[2m, 5m-4]$, and since $(5m-6)-(3m-1) = 2m-5 \geq m + \lceil \frac{m}{2} \rceil - 4$, it follows, in view of Lemma 4.4 with $\gamma = 2m$ that the proof is complete. □

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