An Extension of the Erdős-Ginzburg-Ziv Theorem to Hypergraphs

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Abstract

An *n-set partition* of a sequence S is a collection of n nonempty subsequences of S, pairwise disjoint as sequences, such that every term of S belongs to exactly one of the subsequences, and the terms in each subsequence are all distinct so that they can be considered as sets. For a sequence S, subsequence S' and set T, then $|T \cap S|$ denotes the number of terms x of S with $x \in T$, and $|S|$ denotes the length of S, and $S \setminus S'$ denotes the subsequence of S obtained by deleting all terms in S'. We first prove the following two additive number theory results.

(1) Let S be a finite sequence of elements from an abelian group G. If S has an n-set partition, $A = A_1, \ldots, A_n$, such that

$$|\sum_{i=1}^{n} A_i| \geq \sum_{i=1}^{n} |A_i| - n + 1,$$

then there exists a subsequence S' of S, with length $|S'| \leq \max\{|S| - n + 1,\ 2n\}$, and with an n-set partition, $A' = A'_1, \ldots, A'_n$, such that $|\sum_{i=1}^{n} A'_i| \geq \sum_{i=1}^{n} |A_i| - n + 1$. Furthermore, if $||A_i| - |A_j|| \leq 1$ for all i and j, or if $|A_i| \geq 3$ for all i, then $A'_i \subseteq A_i$.

(2) Let S be a sequence of elements from a finite abelian group G of order m, and suppose there exist $a, b \in G$ such that $|(G \setminus \{a, b\}) \cap S| \leq \lfloor \frac{m}{2} \rfloor$. If $|S| \geq 2m - 1$, then there exists an m-term zero-sum subsequence S' of S with $|\{a\} \cap S'| \geq \lfloor \frac{m}{2} \rfloor$ or $|\{b\} \cap S'| \geq \lfloor \frac{m}{2} \rfloor$.

Let $\mathcal{H}$ be a connected, finite m-uniform hypergraph, and let $f(\mathcal{H})$ (let $f_{zs}(\mathcal{H})$) be the least integer n such that for every 2-coloring (coloring with the elements of the cyclic group $\mathbb{Z}_m$) of the vertices of the complete m-uniform hypergraph $\mathcal{K}_n^m$, there exists a subhypergraph $\mathcal{K}$ isomorphic to $\mathcal{H}$ such that every edge in $\mathcal{K}$ is monochromatic (such that for every edge e in $\mathcal{K}$ the sum of the colors on e is zero). As a corollary to the above theorems, we show that if every subhypergraph $\mathcal{H}'$ of $\mathcal{H}$ contains an edge with at least half of its vertices monovalent in $\mathcal{H}'$, or if $\mathcal{H}$ consists of two intersecting edges, then $f_{zs}(\mathcal{H}) = f(\mathcal{H})$. This extends the Erdős-Ginzburg-Ziv Theorem, which is the case when $\mathcal{H}$ is a single edge.

1 Introduction

Let $(G, +, 0)$ be an abelian group. If $A, B \subseteq G$, then their *sumset*, $A + B$, is the set of all possible pairwise sums, i.e. $\{a + b \mid a \in A, b \in B\}$. If S is a sequence of elements from G, then an *n-set partition* of S is a collection of n nonempty subsequences of S, pairwise disjoint as sequences, such that every term of S belongs to exactly one of the subsequences, and the terms in each subsequence are all distinct. Thus such subsequences can be considered as sets. A sequence is *zero-sum* if the sum of its terms is zero. For a sequence S and set T, we use $|T \cap S|$ to denote the number of terms x of S with $x \in T$. Also, $|S|$ denotes the cardinality of S, if S is a set, and the length of S, if S is a sequence. If S' is a subsequence of S, then $S \setminus S'$ denotes the subsequence of S obtained by deleting all terms in S'.

Let $\mathcal{H}$ be an m-uniform hypergraph. Then the vertex set of $\mathcal{H}$ is denoted $V(\mathcal{H})$, and its edge set is denoted $E(\mathcal{H})$. If $\Delta : V(\mathcal{H}) \to \mathbb{Z}_m$ is a vertex coloring of $\mathcal{H}$ by the cyclic group of order m, then $\mathcal{H}$ is edgewise zero-sum if every $e \in E(\mathcal{H})$ satisfies $\sum_{v \in e} \Delta(v) = 0$. A monovalent vertex is a vertex contained in precisely one edge. Finally, let $\mathcal{K}_n^m$ be the complete m-uniform hypergraph on n vertices.

We begin with the Erdős-Ginzburg-Ziv theorem [8] [1] [20].

Erdős-Ginzburg-Ziv Theorem (EGZ). *Let G be an abelian group of order m, and let S be a sequence of elements from G. If $|S| \geq 2m-1$, then S contains an m-term zero-sum subsequence.*

Observe that if S is a sequence of 0's and 1's from the cyclic group $\mathbb{Z}_m$, then the m-term monochromatic subsequences of S correspond exactly with the m-term zero-sum subsequences. Thus the Erdős-Ginzburg-Ziv Theorem can be thought of as a generalization of the pigeonhole principle for m pigeons and 2 holes. This has allowed several Ramsey-type questions to be generalized by replacing colorings using two elements with colorings using the elements from $\mathbb{Z}_m$, and looking for zero-sum substructures rather than monochromatic ones. If m is chosen to be the size of the particular substructure in question, then the zero-sum Ramsey number always gives an upper bound on the monochromatic Ramsey number. However, in many cases, the two numbers are in fact equal. Such problems are said to zero-sum generalize. Examples include questions that involve looking for a single zero-sum substructure [9] [21] [4], and those that involve looking for several, disjoint substructures that are each individually zero-sum [5] [14]. A survey of related problems can be found in [6]. However, until recently, it was not known if even the two simplest zero-sum Ramsey questions involving nondisjoint structures—namely two individually zero-sum m-term subsequences that share exactly one vertex; and two that share exactly two vertices—would zero-sum generalize. Both these cases were found to zero-sum generalize [2], leaving the question of what other overlapping structures might zero-sum generalize.

Formalizing the above thoughts in the language of hypergraphs, let $f(\mathcal{H})$ (let $f_{zs}(\mathcal{H})$) be the least integer n such that for every 2-coloring (coloring with the elements of $\mathbb{Z}_m$) of the vertices of $\mathcal{K}_n^m$, there exists a subhypergraph $\mathcal{K}$ isomorphic to $\mathcal{H}$ such that every edge e in $\mathcal{K}$ is monochromatic, i.e. has all its vertices of the same color (such that for every edge e in $\mathcal{K}$ the sum of the colors on e is zero). It is clear from the pigeonhole principle that $f(\mathcal{H}) \leq 2|V(\mathcal{H})|-1$, with equality holding if H is connected. Under this phrasing, the Erdős-Ginzburg-Ziv Theorem becomes the statement that if $\mathcal{H}$ is a single edge, then $f_{zs}(\mathcal{H}) = f(\mathcal{H})$, i.e. $\mathcal{H}$ edgewise zero-sum generalizes.

In this paper, we make the first tentative step towards classifying those hypergraphs that edgewise zero-sum generalize, by proving the following.

Theorem 1.1. *Let $\mathcal{H}$ be a connected, finite m-uniform hypergraph. If every subhypergraph $\mathcal{H}'$ of $\mathcal{H}$ contains an edge with at least half of its vertices monovalent in $\mathcal{H}'$, then $\mathcal{H}$ edgewise zero-sum generalizes.*

Theorem 1.2. *If $\mathcal{H}$ is a hypergraph that consists of two intersecting m-sets, then $\mathcal{H}$ edgewise zero-sum generalizes.*

As will later be seen in Section 5, there exist m-uniform hypergraphs with every edge having at least $\lceil \frac{m}{2} \rceil - 2$ of its vertices monovalent, but which do not edgewise zero-sum generalize. Hence the bound on the number of monovalent vertices in Theorem 1.1 can be improved at most by one, after which more refined properties must be sought to determine if $\mathcal{H}$ edgewise zero-sum generalizes.

We will derive Theorems 1.1 and 1.2 as simple corollaries to a recent theorem in [13], referred to in this paper as Theorem 2.1, and the following two general theorems from additive number theory, which we prove in Sections 3 and 4, respectively. Theorem 1.3 shows that we can drain elements out of an n-set partition while leaving the sumset of the set partition relatively unaffected—an ability that can be quite useful in zero-sum applications as it frees up additional terms that might not be available for further use otherwise. Theorem 1.4 is a refinement of the Erdős-Ginzburg-Ziv Theorem that shows in a mostly two color sequence of length $2m-1$, there is a mostly monochromatic m-term zero-sum subsequence. The proof of Theorem 1.3 makes use of recent machinery [11] for the Kemperman Structure Theorem (KST) for critical pairs (i.e. pairs of finite subsets (A, B) of an abelian group with $|A+B| \leq |A|+|B|-1$) [Theorems 5.1 and 3.4 and comments on pp. 81–82, 16], while the proof of Theorem 1.4 makes use of a method first introduced by Gao and Hamidoune [10].

Theorem 1.3. *Let S be a finite sequence of elements from an abelian group G. If S has an n-set partition, $A = A_1, \ldots, A_n$, such that*

$$|\sum_{i=1}^{n} A_i| \geq \sum_{i=1}^{n} |A_i| - n + 1, \tag{1}$$

then there exists a subsequence S' of S, with length $|S'| \leq \max\{|S|-n+1,\ 2n\}$, and with an n-set partition, $A' = A'_1, \ldots, A'_n$, such that $|\sum\limits_{i=1}^{n} A'_i| \geq \sum\limits_{i=1}^{n} |A_i| - n + 1$. Furthermore, if $||A_i| - |A_j|| \leq 1$ for all i and j, or if $|A_i| \geq 3$ for all i, then $A'_i \subseteq A_i$.

Theorem 1.4. *Let S be a sequence of elements from a finite abelian group G of order m, and suppose there exist $a,\ b \in G$ such that $|(G \setminus \{a, b\}) \cap S| \leq \lfloor \frac{m}{2} \rfloor$. If $|S| \geq 2m-1$, then there exists an m-term zero-sum subsequence S' of S with $|\{a\} \cap S'| \geq \lfloor \frac{m}{2} \rfloor$ or $|\{b\} \cap S'| \geq \lfloor \frac{m}{2} \rfloor$.*

Note that the sequence $S = (\underbrace{0, \ldots, 0}_{n}, \underbrace{1, \ldots, 1}_{n}, \underbrace{2, \ldots, 2}_{n'})$, where $n' \leq n$ and $G = \mathbb{Z}_m$, shows that the bound on $|S'|$ in Theorem 1.3 is tight for $|S| \leq 3n$. The sequence $S = (\underbrace{0, \ldots, 0}_{m-1}, \underbrace{1, \ldots, 1}_{m-1}, \lceil \frac{m}{2} \rceil)$ with $G = \mathbb{Z}_m$ shows that the lower bound $\lfloor \frac{m}{2} \rfloor$ in Theorem 1.4 is also tight, although the theorem likely remains true under a less restrictive condition than $|(G \setminus \{a, b\}) \cap S| \leq \lfloor \frac{m}{2} \rfloor$.

2 Preliminaries

Let $A,\ B \subseteq G$, where G is an abelian group. We denote by $\nu_c(A, B)$ the number of representations of $c = a + b$ with $a \in A$ and $b \in B$. We denote by $\eta_b(A, B)$ the number of $c \in A + b$ such that $\nu_c(A, B) = 1$. A set $A \subseteq G$ is said to be H_a*-periodic*, if it is the union of H_a-cosets for some nontrivial subgroup H_a of G, and otherwise, A is called *aperiodic*. We say that A is *maximally* H_a-periodic, if A is H_a-periodic, and H_a is the maximal subgroup for which A is periodic; in this case, $H_a = \{x \in G \mid x + A = A\}$, and H_a is sometimes referred to as the *stabilizer* of A. If $A+B$ is H_a-periodic, then an H_a*-hole* of A (where the subgroup H_a is usually understood) is an element $\alpha \in (A + H_a) \setminus A$. We will use $\phi_a : G \to G/H_a$ to denote the natural homomorphism.

We begin by stating Kneser's Theorem [18] [16] [19] [17] [20] [15]. The case with m prime is known as the Cauchy-Davenport Theorem [7].

Kneser's Theorem. *Let G be an abelian group, and let $A_1, A_2, \ldots, A_n$ be a collection of finite, nonempty subsets of G. If $\sum\limits_{i=1}^{n} A_i$ is maximally H_a-periodic, then*

$$\left|\sum_{i=1}^{n} \phi_a(A_i)\right| \geq \sum_{i=1}^{n} |\phi_a(A_i)| - n + 1,$$

and otherwise the above inequality holds with ϕ_a the identity.

Note that if A is maximally H_a-periodic, then $\phi_a(A)$ is aperiodic. Also, observe that if $A+B$ is maximally H_a-periodic and $\rho = |A+H_a| - |A| + |B+H_a| - |B|$ is the number of holes in A and B, then Kneser's Theorem implies $|A+B| \geq |A| + |B| - |H_a| + \rho$. Consequently, if either A or B contains a unique element from some H_a-coset, then $|A+B| \geq |A| + |B| - 1$. More generally, if ρ is the total number of holes in the A_i, then $\left|\sum_{i=1}^{n} A_i\right| \geq \sum_{i=1}^{n} |A_i| - (n-1)|H_a| + \rho$. The following is a recent composite analog of the Cauchy-Davenport Theorem [13] [12].

Theorem 2.1. *Let S be a sequence of elements from an abelian group G of order m with an n-set partition $P = P_1, \ldots, P_n$, and let p be the smallest prime divisor of m. Then either:*

(i) there exists an n-set partition $A = A_1, A_2, \ldots, A_n$ of S such that:

$$|\sum_{i=1}^{n} A_i| \geq \min\{m,\ (n+1)p,\ |S| - n + 1\};$$

furthermore, if $n' \geq \frac{m}{p} - 1$ is an integer such that P has at least $n - n'$ cardinality one sets and if $|S| \geq n + \frac{m}{p} + p - 3$, then we may assume there are at least $n - n'$ cardinality one sets in A, or

(ii) (a) there exists $\alpha \in G$ and a nontrivial proper subgroup H_a of index a such that all but at most $a-2$ terms of S are from the coset $\alpha + H_a$; and (b) there exists an n-set partition $A_1, A_2, \ldots, A_n$ of the subsequence of S consisting of terms from $\alpha + H_a$ such that $\sum_{i=1}^{n} A_i = n\alpha + H_a$.

The following two simple propositions are often helpful when using n-set partitions, and proofs can be found in [3]. In [3], Proposition 2.2 was stated only in the case $|B| = 1$ and $r' = r$, but the proof given there also proves the more general statement given here.

Proposition 2.1. *A sequence S has an n-set partition A if and only if the multiplicity of each element in S is at most n and $|S| \geq n$. Furthermore, a sequence S with an n-set partition has an n-set partition $A' = A_1, \ldots, A_n$ such that $||A_i| - |A_j|| \leq 1$ for all i and j satisfying $1 \leq i \leq j \leq n$.*

Proposition 2.2. *Let S be a finite sequence of elements from an abelian group G, let B be a finite, nonempty subset of G, and let $A = A_1, \ldots, A_n$ be an n-set partition of S, where $|B + \sum_{i=1}^{n} A_i| - |B| + 1 = r$, and $\max_i\{|B + A_i| - |B| + 1\} = s$. Furthermore, let $a_1, \ldots, a_n$ be a subsequence of S such that $a_i \in A_i$ for $i = 1, \ldots, n$, and let r' be an integer with $1 \leq r' \leq r$.*

(i) There exists a subsequence S' of S and an n'-set partition $A' = A'_1, \ldots, A'_{n'}$ of S', which is a subsequence of the n-set partition A, such that $n' \leq r - s + 1$ and $|B + \sum_{i=1}^{n'} A'_i| = |B + \sum_{i=1}^{n} A_i|$.

(ii) There exists a subsequence S' of S of length at most $n + r' - 1$, and an n-set partition $A' = A'_1, \ldots, A'_n$ of S', where $A'_i \subseteq A_i$ for $i = 1, \ldots, n$, such that $|B + \sum_{i=1}^{n} A'_i| = |B| - 1 + r'$. Furthermore, $a_i \in A'_i$ for $i = 1, \ldots, n$.

The following lemma was originally used in the proof of Kneser's Theorem [17] [20] [18].

Kneser Lemma. *Let C_0 be a finite subset of an abelian group. If $C_0 = C_1 \cup C_2$ with $C_i \neq C_0$ $(i = 1, 2)$, then $\min_{i=1,2}\{|C_i|+|H_{k_i}|\} \leq |C_0|+|H_{k_0}|$, where H_{k_i} is the trivial group if C_i is aperiodic, and otherwise H_{k_i} is the maximal group for which C_i is H_{k_i}-periodic $(i = 0, 1, 2)$.*

We will also need the following [17].

Theorem 2.2. *Let G be a group, and let A, $B \subseteq G$ be finite subsets. If $|A+B| = |A|+|B|-\rho$, then $\nu_c(A, B) \geq \rho$ for all $c \in A + B$.*

Finally, the following elementary result will be used [20].

Theorem 2.3. *Let G be a finite abelian group, and let A, $B \subseteq G$. If $|A| + |B| > |G|$, then $A + B = G$.*

3 A Draining Theorem for Set Partitions

Let G be an abelian group, and let H_a be a nontrivial subgroup. If $A \subseteq G$, then a *quasi-periodic decomposition* of A with *quasi-period* H_a is a partition $A = A_1 \cup A_0$ of A into two disjoint (each possibly empty) subsets such that A_1 is H_a-periodic or empty and A_0 is a subset of an H_a-coset. A set $A \subseteq G$ is *quasi-periodic* if A has a quasi-periodic decomposition $A = A_1 \cup A_0$ with A_1 nonempty. Such a decomposition is *reduced* if A_0 is not quasi-periodic. Quasi-periodic decompositions play an important role in the KST description of critical pairs. Observe that if A is finite and has a quasi-periodic decomposition $A_1 \cup A_0$, then A has a reduced quasi-periodic decomposition $A'_1 \cup A'_0$ with $A'_0 \subseteq A_0$, and that an arithmetic progressions with difference d and at most $|\langle d\rangle| - 2$ terms is an example of a non-quasi-periodic set. A *punctured periodic set*, i.e. a set A for which there exists $\alpha \in G \setminus A$ such that $A \cup \{\alpha\}$ is maximally H-periodic, has a reduced quasi-periodic decomposition for each prime order subgroup of H. However, quasi-periodic decompositions are otherwise canonical, as seen by the following proposition [11].

Proposition 3.1. *If $A_1 \cup A_0$ and $A'_1 \cup A'_0$ are both reduced quasi-periodic decompositions of a subset A of an abelian group G, with A_1 maximally H-periodic and A'_1 maximally L-periodic, then either (i) $A_1 = A'_1$ and $A_0 = A'_0$ or (ii) $H \cap L$ is trivial, $A_0 \cap A'_0 = \emptyset$, $|H|$ and $|L|$ are prime, and there exists $\alpha \in G \setminus A$ such that $A_0 \cup \{\alpha\}$ is an H-coset, $A'_0 \cup \{\alpha\}$ is an L-coset, and $A \cup \{\alpha\}$ is $(H + L)$-periodic.*

In the case of $n = 2$, we have the following versions of Theorem 1.3 [11].

Theorem 3.1. *Let G be an abelian group, and let A, $B \subseteq G$ be finite subsets such that $|A| \geq 2$, and $|B| \geq 3$. If $|A + B| \geq |A| + |B| - 1$, then either:*

(i) *there exists* $b \in B$ *such that* $|A + (B \setminus \{b\})| \geq |A| + |B| - 1$, *or*

(ii) (a) $|A + B| = |A| + |B| - 1$, (b) *there exists* $a \in A$ *such that* $A \setminus \{a\}$ *is* H_a*-periodic, and* (c) *there exists* $\alpha \in G$ *such that* $B \subseteq \alpha + H_a$.

Theorem 3.2. *Let* G *be an abelian group, and let* A, B, $C_1, \ldots, C_r \subseteq G$ *be finite subsets with* $|B| \geq 3$. *If* $|A + B| > |A| + |B| - 1$, $|A + B + \sum_{i=1}^{r} C_i| \geq |A| + |B| + \sum_{i=1}^{r} |C_i| - (r + 2) + 1$, *and* $|A + \sum_{i=1}^{r} C_i| \geq |A| + \sum_{i=1}^{r} |C_i| - (r + 1) + 1$, *then there exists* $b \in B$ *such that* $|A + (B \setminus \{b\})| \geq |A| + |B| - 1$ *and* $|A + (B \setminus \{b\}) + \sum_{i=1}^{r} C_i| \geq |A| + |B| + \sum_{i=1}^{r} |C_i| - (r + 2) + 1$.

We note that conclusion (ii) of Theorem 3.1 implies both that $|A + (B \setminus \{b\})| \geq |A| + |B| - 2$ for all $b \in B$, and that $|A| > |B|$, so that by interchanging the roles of A and B we can be assured that (i) will hold. We can now begin the proof of Theorem 1.3.

Proof Theorem 1.3. We may assume $|S| \geq 2n + 1$ and $n \geq 2$, else the theorem is trivial. We may also assume $n \geq 3$, since the $n = 2$ case follows from Theorem 3.1. Let $|S| = sn + r$, where $s \geq 2$ and $0 \leq r < n$. If neither of the conditions of the furthermore part of Theorem 1.3 hold, then we may w.l.o.g. assume that A was chosen from all n-set partitions of S that satisfy (1) so that the cardinality s' of the minimal cardinality set A_i in A is maximal, and such that, subject to prior conditions, the number of terms A_i in A with cardinality s' is minimal. Re-index so that the cardinalities of the A_i are nondecreasing, and assume that $|A_i| \geq s + 2$ for $i > k_2$, and that $|A_i| \leq \min\{2, s - 1\}$ for $i < k_1$.

The remainder of the proof is divided into two cases. The first handles the case when either all sets A_i have cardinality at least three, or all have cardinality equal to two or three. Under these conditions, we show in Case 1b that we can inductively remove terms from the sets A_i one by one unless highly restrictive conditions occur. Under these restrictive conditions, we show in Case 1a that we can complete the removal of the remaining terms in one swipe. We note that the complexity of the induction statement in Case 1b arises from the exceptional case in Theorem 3.1, and that without this problem the induction would go through quite smoothly. Finally, Case 2 handles the case when the set-partition A can't be reduced to one satisfying the conditions of Case 1. In this case, a similar argument to that of Case 1a works quite simply provided the Cauchy-Davenport bound does not hold for every subsequence of A. Thus the majority of Case 2 is spent showing that it is quite difficult for a set-partition A to satisfy Cauchy-Davenport everywhere and not be reducible to a set partition either with a larger minimal cardinality set or with a fewer number of minimal cardinality sets.

Case 1a: Suppose that $k_1 = 1$, and if $s = 2$ that $k_2 = n$ (note if either of the conditions of the furthermore part of Theorem 1.3 hold, then this will be the case). Further suppose that, allowing re-indexing, there exists an n-set partition, $A' = A'_1, \ldots, A'_n$, of a subsequence S' of S, and an integer l with $2 \le l \le n$, such that

$$|\sum_{i=1}^{n} A'_i| \ge \sum_{i=1}^{n} |A_i| - n + 1, \tag{2}$$

$A'_i \subseteq A_i$, $\sum\limits_{i=1}^{l} A'_i$ is maximally H_a-periodic, $\sum\limits_{i=1}^{l} |A'_i| = |A_1| + \sum\limits_{i=2}^{l} \max\{2,\, |A_i| - 1\}$, $|A_1| = \min\limits_i\{|A_i|\}$, $A'_i = A_i$ for $i > l$, $|A'_l| \ge \max\{2,\, |A_l| - 1\}$,

$$|\sum_{i=1}^{l-1} A'_i| \ge \sum_{i=1}^{l-1} |A_i| - (l-1) + 1, \tag{3}$$

and

$$|\sum_{i=1}^{l} A'_i| < \sum_{i=1}^{l} |A_i| - l + 1. \tag{4}$$

Let b be the integer such that

$$b|H_a| < \sum_{i=1}^{n} |A_i| - n + 1 \le (b+1)|H_a|, \tag{5}$$

let ρ be the integer such that

$$|\sum_{i=1}^{l} A'_i| = |\sum_{i=1}^{l-1} A'_i| + |A'_l| - 1 - \rho, \tag{6}$$

let $s_2 = \sum\limits_{i=l+1}^{n} |A_i|$, let $s_1 = \sum\limits_{i=1}^{l} |A_i|$, and let $s'_1 = \sum\limits_{i=1}^{l} |A'_i|$.

Since $|A'_l| \ge |A_l| - 1$ and since $A'_l \subseteq A_l$, then in view (3), (4) and (6), it follows that $0 \le \rho \le |A_l| - 1$. Furthermore, in view of Theorem 2.2, it follows that there exists a proper subset $T \subseteq A'_l$ of cardinality ρ such that $\sum\limits_{i=1}^{l-1} A'_i + (A'_l \setminus T) = \sum\limits_{i=1}^{l} A'_i$.

Let S'' be a minimal length subsequence of the terms of S' partitioned by the $A'_i = A_i$ where $i \ge l+1$, with an $(n-l)$-set partition, $B' = B_1, \ldots, B_{n-l}$, such that $|\sum\limits_{i=1}^{l} \phi_a(A'_i) + \sum\limits_{i=1}^{n-l} \phi_a(B_i)| \ge b+1$ and $B_i \subseteq A_{i+l}$ (since $\sum\limits_{i=1}^{l} A'_i$ is H_a-periodic, such a subsequence exists by (2) and (5)). Since $\sum\limits_{i=1}^{l} |A'_i| = |A_1| + \sum\limits_{i=2}^{l} \max\{2,\, |A_i| - 1\}$, since $|A_1| = \min\limits_i\{|A_i|\}$, since $A'_i \subseteq A_i$, since $k_1 = 1$, since $k_2 = n$ if $s = 2$, and since $\sum\limits_{i=1}^{l} A'_i$ is H_a-periodic, it follows in view of (5) and the conclusion of the last paragraph that the proof will be complete unless

$$s_2 - s'_2 \le n - l - 1 - \rho, \tag{7}$$

where $s'_2 = |S''|$. Hence $l < n$. From the minimality of S'' it follows that $|B_j| = |\phi_a(B_j)|$, and furthermore, for $x \in B_j$ with $|B_j| \ge 2$, that

$$\eta_{\phi_a(x)}\left(\sum_{i=1}^{l} \phi_a(A'_i) + \sum_{i=1}^{j-1} \phi_a(B_i),\, \phi_a(B_j)\right) \ge 1. \tag{8}$$

Hence, since $A'_i \subseteq A_i$, since $\sum_{i=1}^{l} A'_i$ is H_a-periodic, and since $|A'_l| \geq |A_l| - 1$, it follows, in view of (8), (3), (6) and (5), that we can remove an element from S'' contained in the set B_j with greatest index such that $|B_j| \geq 2$ (since $k_1 = 1$ and $A'_i \subseteq A_i$, such a set exists in view of (7)) and contradict the minimality of S'' unless

$$(s'_2 - (n-l) - 1)|H_a| \leq s_2 - (n-l) + \rho. \tag{9}$$

Using the estimate $|H_a| \geq 2$, it follows from (9) that

$$s'_2 \leq (s_2 - s'_2) + \rho + (n-l) + 2. \tag{10}$$

However, (10) and (7) imply that

$$s'_2 \leq 2(n-l) + 1. \tag{11}$$

Hence the proof is complete unless $\rho = 0$ and equality holds in (11), which can only occur if $|H_a| = 2$.

If $|A'_l| \geq 3$, then since $\rho = 0$, and since $\sum_{i=1}^{l} A'_i$ is maximally H_a-periodic, it follows from (6), Proposition 3.1 and Theorem 3.1 that either we can remove an additional element from A'_l leaving the sumset unchanged, whence the proof is complete, or else A'_l is maximally $H_{a'}$-periodic with $H_{a'} \leq H_a$, whence since $|H_a| = 2$ it follows that A'_l is maximally H_a-periodic. If $|A'_l| = 2$, then since $\rho = 0$, and since $\sum_{i=1}^{l} A'_i$ is maximally H_a-periodic, it follows from (6) and Kneser's Theorem that $|\phi_a(A'_l)| = 1$, whence since $|H_a| = 2$ it follows that A'_l is H_a-periodic. Thus regardless of the cardinality of A'_l we may assume A'_l is H_a-periodic. Hence it follows that there does not exist a set A'_j with $j < l$ and $|\phi_a(A'_j)| < |A'_j|$, since otherwise we can remove an additional element from A'_j leaving the sumset unchanged and completing the proof. Hence, since $\sum_{i=1}^{l} A'_i$ is maximally H_a-periodic, and since $|H_a| = 2$, it follows in view of Kneser's Theorem and (4) that $s_1 - l \geq |\sum_{i=1}^{l} A'_i| \geq 2(s'_1 - l + 1 - |A'_l|) + |A'_l|$. Since $A'_i \subseteq A_i$, since $k_1 = 1$, and since $s'_1 = \sum_{i=1}^{l} |A'_i| = |A_1| + \sum_{i=2}^{l} \max\{2, |A_i| - 1\}$, it follows that

$$s_1 \leq s'_1 + l - 1. \tag{12}$$

Hence, since $s_1 - l \geq 2(s'_1 - l + 1 - |A'_l|) + |A'_l|$, it follows that $s'_1 \leq 2l - 3 + |A'_l|$. Hence, if $|A'_l| = 2$, then in view of (11) it follows that the proof is complete. So we may assume $|A'_l| > 2$. Thus, since A'_l is H_a-periodic, and since $|H_a| = 2$, it follows that $|A'_l| \geq 4$. Hence, since $k_2 = n$ if $s = 2$, and since $A'_l \subseteq A_l$, it follows that $s \geq 3$. Since $s'_1 \leq 2l - 3 + |A'_l|$, it follows that $\sum_{i=1}^{l-1} |A'_i| \leq 2(l-1) - 1$. Consequently, since $s \geq 3$, since $k_1 = 1$, and since $A'_i \subseteq A_i$, it follows that $s_1 \geq s'_1 + l$, a contradiction to (12).

Case 1b: Suppose that $k_1 = 1$, and if $s = 2$ that $k_2 = n$. We proceed by induction on a parameter l, with $1 \le l \le n$, as follows. Inductively assume, passing from $l-1$ to l, that (allowing re-indexing) we can remove elements from the sets A_i with $i \le l-1$, yielding new, nonempty sets A'_i, such that $\sum_{i=1}^{l-1}|A'_i| = |A_1| + \sum_{i=2}^{l-1}\max\{2, |A_i|-1\}$, such that $|A_1| = \min_i\{|A_i|\}$, such that (2) and (3) hold with $A'_i = A_i$ for $i > l-1$, and such that $|A'_{l-1}| \ge \max\{2, |A_{l-1}|-1\}$; furthermore, if $l-1 > 1$, if equality holds in (3), if

$$\sum_{i=1}^{l-1} A'_i = H \cup \{b\}, \tag{13}$$

where H is maximally H_a-periodic and $b \notin H$, and if $|H_a| > 2$, then

$$|\sum_{i=1}^{(l-1)-1} A'_i| \ge \sum_{i=1}^{(l-1)-1} |A_i| - ((l-1)-1) + \epsilon, \tag{14}$$

where $\epsilon = 0$ if $|A'_{l-1}| > 3$ and $|A'_{l-1}| = |A_{l-1}|$, and $\epsilon = 1$ if $|A'_{l-1}| \le 3$ or $|A'_{l-1}| = |A_{l-1}| - 1$. The case $l = 1$ is trivial. Note also that the $l = n$ case completes the proof, so that Case 1 will be complete once the induction is completed. Further note that (3) with parameter $l-1$ implies (14) with parameter l (in place of $(l-1)$).

Suppose there exists a set A_r with $r > l-1$ such that $|\sum_{i=1}^{l-1} A'_i + A_r| < \sum_{i=1}^{l-1}|A_i| + |A_r| - l + 1$. Hence from (3) it follows that $|\sum_{i=1}^{l-1} A'_i + A_r| < |\sum_{i=1}^{l-1} A'_i| + |A_r| - 1$, whence from Kneser's Theorem it follows that $\sum_{i=1}^{l-1} A'_i + A_r$ is maximally H_a-periodic, and from Theorem 2.2 it follows (for $|A_r| \ge 3$) that we can remove some element x from A_r to yield a new set A'_r, such that $\sum_{i=1}^{l-1} A'_i + A_r = \sum_{i=1}^{l-1} A'_i + A'_r$. Hence, after re-indexing, the conditions of Case 1a are met, and so we may assume $|\sum_{i=1}^{l-1} A'_i + A_r| \ge \sum_{i=1}^{l-1}|A_i| + |A_r| - l + 1$. Consequently, we may assume $|A_r| > 2$ for $r > l-1$, else the induction is complete.

Suppose there exists a set A_r with $r > l-1$ such that $|\sum_{i=1}^{l-1} A'_i + A_r| < |\sum_{i=1}^{l-1} A'_i| + |A_r| - 1$. Then from Theorem 2.2 it follows that we can remove some element x from A_r to yield a new set A'_r such that $\sum_{i=1}^{l-1} A'_i + A_r = \sum_{i=1}^{l-1} A'_i + A'_r$. If $|\sum_{i=1}^{l-1} A'_i + A_r| \ge \sum_{i=1}^{l-1}|A_i| + |A_r| - l + 1$, then the induction is complete, and otherwise we reduce to the conditions of the previous paragraph. So we may assume that $|\sum_{i=1}^{l-1} A'_i + A_r| \ge |\sum_{i=1}^{l-1} A'_i| + |A_r| - 1$ for all $r > l-1$.

Suppose that the inequality in (3) is strict. Suppose further that $|\sum_{i=1}^{l-1} A'_i + \sum_{i=l+1}^{n} A_i| < |\sum_{i=1}^{l-1} A'_i| + \sum_{i=l+1}^{n} |A_i| - (n-l+1) + 1$. Hence in view of Theorem 2.2 it follows that there exists a set A_r with $r \ge l+1$ such that $\sum_{i=1}^{l-1} A'_i + \sum_{i=l+1}^{n} A_i = \sum_{i=1}^{l-1} A'_i + \sum_{\substack{i=l+1\\ i\ne r}}^{n} A_i + (A_r \setminus \{x\})$ for all $x \in A_r$. In view of

Theorem 3.1 and the conclusion of the last paragraph, it follows that there exists $x \in A_r$ such that $|\sum_{i=1}^{l-1} A'_i + (A_r \setminus \{x\})| \geq |\sum_{i=1}^{l-1} A'_i| + |A_r| - 2$. Hence since the inequality in (3) is strict, and since $\sum_{i=1}^{l-1} A'_i + \sum_{i=l+1}^{n} A_i = \sum_{i=1}^{l-1} A'_i + \sum_{\substack{i=l+1 \\ i \neq r}}^{n} A_i + (A_r \setminus \{x\})$, it follows that the induction is complete letting $A'_l = A_r \setminus \{x\}$. So we may assume $|\sum_{i=1}^{l-1} A'_i + \sum_{i=l+1}^{n} A_i| \geq |\sum_{i=1}^{l-1} A'_i| + \sum_{i=l+1}^{n} |A_i| - (n-l+1) + 1$.

Since the inequality in (3) is strict, and in view of the conclusion of the third paragraph of Case 1b (with $r = l$), then it follows from Theorem 2.2 that $|\sum_{i=1}^{l-1} A'_i + (A_l \setminus \{x\})| \geq \sum_{i=1}^{l} |A_i| - l + 1$, for all but at most one (say x_0) $x \in A_l$. Hence the induction is complete letting $A'_l = A_l \setminus \{x\}$, with $x \in A_l$ and $x \neq x_0$, unless $|\sum_{i=1}^{l-1} A'_i + (A_l \setminus \{x\}) + \sum_{i=l+1}^{n} A_i| < \sum_{i=1}^{n} |A_i| - n + 1$. Hence, in view of strict inequality in (3) and the conclusion of the last paragraph, it follows that $|\sum_{i=1}^{l-1} A'_i + (A_l \setminus \{x\}) + \sum_{i=l+1}^{n} A_i| < |\sum_{i=1}^{l-1} A'_i + \sum_{i=l+1}^{n} A_i| + |(A_l \setminus \{x\})| - 1$, whence in view of Theorem 2.2 it follows that the induction is complete by letting $A'_l = A_l \setminus \{x'\}$ for any $x' \in A_l \setminus \{x, x_0\}$. So (since $|A_l| \geq 3$) we may assume that equality holds in (3).

Suppose there exists a set A_r with $r > l - 1$ such that $|\sum_{i=1}^{l-1} A'_i + A_r| = |\sum_{i=1}^{l-1} A'_i| + |A_r| - 1$. Hence, since $|A'_1| \leq |A_1| \leq |A_r|$, and since $|A_r| \geq 3$, then from Theorem 3.1 it follows that either the induction is complete or else (13) holds with $|H_a| > 2$, $A_r \subseteq \alpha + H_a$ for some $\alpha \in G$, and $l > 2$. Hence, since equality holds in (3), it follows by inductive assumption that (14) holds. Hence, since equality holds in (3), and since $|A'_{l-1}| \geq |A_{l-1}| - 1$, it follows that there exists a subset $H' \subset H \cup \{b\}$ with cardinality at most $|A'_{l-1}| + 1 - \epsilon$, such that $\sum_{i=1}^{l-2} A'_i = \beta + (H \cup \{b\}) \setminus H'$, for some $\beta \in G$.

Suppose $|H_a| > |A'_{l-1}| + 2 - \epsilon$. Hence, since H is H_a-periodic, and since $|H'| \leq |A'_{l-1}| + 1 - \epsilon$, it follows that if an H_a-coset $\gamma + H_a$ contains at least two elements of $\sum_{i=1}^{l-1} A'_i = H \cup \{b\}$, then the H_a-coset $(\beta + \gamma) + H_a$ will contain at least two elements of $\sum_{i=1}^{l-2} A'_i$. Hence, since $|A'_{l-1}| \geq 2$, it follows from (13) that $|\phi_a(A'_{l-1})| > 1$ and that $b \notin H'$, since if the contrary holds in either case, then $H \cup \{b\}$ will contain at least two elements from every H_a-coset that intersects $H \cup \{b\}$, a contradiction. Hence from the conclusions of the last two sentences it follows that $\phi_a(\sum_{i=1}^{l-2} A'_i) = \phi_a(\sum_{i=1}^{l-1} A'_i)$, whence since $|\phi_a(A'_{l-1})| > 1$, it follows in view of Theorem 2.2 applied modulo H_a that $\nu_{\phi_a(b)}(\sum_{i=1}^{l-2} \phi_a(A'_i), \phi_a(A'_{l-1})) \geq 2$. Hence there are two elements (say) $c, d \in \sum_{i=1}^{l-2} A'_i$, that are distinct modulo H_a, and each of which can be summed with some element of A'_{l-1} to give us an element from the coset $b + H_a$. Consequently, if the coset class represented by c has at least x elements contained in $\sum_{i=1}^{l-2} A'_i$, then any coset class of b must also contain at least x elements in $\sum_{i=1}^{l-1} A'_i$. Likewise for d. However, by (13) we know that b is the unique element from its H_a-coset in $\sum_{i=1}^{l-1} A'_i$, and thus by the previous two sentences both c and d must be the unique element from their coset class in $\sum_{i=1}^{l-2} A'_i$. However, it follows from the second sentence of this paragraph that if a coset class contained at least two elements in $\sum_{i=1}^{l-1} A'_i$, then the corresponding (up to translation) coset class of $\sum_{i=1}^{l-2} A'_i$ must also contain at least two elements. Since this is not the case for the two distinct coset classes c and d, it follows that there must be two distinct coset classes with a unique element in $\sum_{i=1}^{l-1} A'_i$, which contradicts (13). So we may assume $|H_a| \leq |A'_{l-1}| + 2 - \epsilon$.

Hence, since $|A_r| \geq 3$ and since A_r is a subset of an H_a-coset, it follows that

$$3 \leq |A_r| \leq |H_a| \leq |A'_{l-1}| + 2 - \epsilon. \tag{15}$$

Let $x \in A'_{l-1}$. If $\sum_{i=1}^{l-2} A'_i + (A'_{l-1} \setminus \{x\}) = \sum_{i=1}^{l-2} A'_i + A'_{l-1}$, then the induction will be complete by letting $A'_{l-1} = A'_{l-1} \setminus \{x\}$ and letting $A'_l = A_r$. Hence $\eta_x(\sum_{i=1}^{l-2} A'_i, A'_{l-1}) \geq 1$ for all $x \in A'_{l-1}$.

Suppose $\eta_{x_i}(\sum_{i=1}^{l-2} A'_i, A'_{l-1}) = 1$ holds for at least two distinct x_1, $x_2 \in A'_{l-1}$. Hence for one of these x_i, say x_1, it follows from (13) that

$$\left|\phi_a\left(\sum_{i=1}^{l-2} A'_i + (A'_{l-1} \setminus \{x_1\})\right)\right| = \left|\phi_a\left(\sum_{i=1}^{l-1} A'_i\right)\right|, \tag{16}$$

whence, since $|A_r| \geq 3$, since A_r is a subset of an H_a-coset, and since $\eta_{x_1}(\sum_{i=1}^{l-2} A'_i, A'_{l-1}) = 1$, it follows from (13) and from Theorem 2.3 that $\sum_{i=1}^{l-2} A'_i + (A'_{l-1} \setminus \{x_1\}) + A_r = \sum_{i=1}^{l-1} A'_i + A_r$, whence the induction is complete for $|A_r| > 3$ by letting $A'_{l-1} = A'_{l-1} \setminus \{x_1\}$ and letting $A'_l = A_r$. So assume $|A_r| = 3$. Hence, since A_r is a subset of an H_a-coset, it follows in view of (13) and (16) that $\left(\sum_{i=1}^{l-2} A'_i + (A'_{l-1} \setminus \{x_1\})\right) + A_r$ has a quasi-periodic decomposition $B_1 \cup B_0$ with $|B_0| = 3$. Hence, in view of Proposition 3.1, it follows that $\left(\sum_{i=1}^{l-2} A'_i + (A'_{l-1} \setminus \{x_1\})\right) + A_r$ cannot have a reduced quasi-periodic decomposition $B'_1 \cup B'_0$ where $|B'_0| = 1$ and B'_1 is maximally $H_{a'}$-periodic with $|H_{a'}| > 2$, since if that were the case, then from the comments from the beginning of Section 3, it would follow from the uniqueness of $B'_1 \cup B'_0$ that $B'_0 \subseteq B_0$ and that $B_0 \setminus B'_0$ was $H_{a'}$-periodic, contradicting that $|B_0 \setminus B'_0| = 2 < |H_{a'}|$. Hence (13) cannot hold for $\left(\sum_{i=1}^{l-2} A'_i + (A'_{l-1} \setminus \{x_1\})\right) + A_r$ with $|H_a| > 2$. Thus, since $\sum_{i=1}^{l-2} A'_i + (A'_{l-1} \setminus \{x_1\}) + A_r = \sum_{i=1}^{l-1} A'_i + A_r$, it follows that the induction will be complete by letting $A'_{l-1} = A'_{l-1} \setminus \{x_1\}$ and letting $A'_l = A_r$. So we may assume that $\eta_x(\sum_{i=1}^{l-2} A'_i, A'_{l-1}) \geq 2$ for all but at most one $x \in A'_{l-1}$.

Hence from (14) it follows that

$$|\sum_{i=1}^{l-1} A'_i| \geq \sum_{i=1}^{l-2} |A_i| - (l-2) + \epsilon + 2(|A'_{l-1}| - 1), \tag{17}$$

which, from the definition of ϵ, and since $|A'_{l-1}| \geq \max\{2, |A_{l-1}| - 1\}$, contradicts that equality holds in (3) unless $|A'_{l-1}| = 2$ and equality holds in (17), whence it follows that $\eta_{x_i}(\sum_{i=1}^{l-2} A'_i, A'_{l-1}) \leq 2$ for both $x_1, x_2 \in A'_{l-1}$. Since $|A'_{l-1}| = 2$, implying $\epsilon = 1$ by induction hypothesis, it follows in view of (15) that $|H_a| = 3$. Hence, since $|A'_{l-1}| = 2$, since $\eta_{x_i}(\sum_{i=1}^{l-2} A'_i, A'_{l-1}) \leq 2$, and in view of (13), it follows for at least one of x_1 and x_2, say x_1, that (16) holds. Hence, since A_r is a subset of an H_a-coset, since $|A_r| \geq 3$, and since $|H_a| = 3$, it follows that A_r is an H_a-coset, that $\sum_{i=1}^{l-1} A'_i + A_r = \sum_{i=1}^{l-2} A'_i + (A'_{l-1} \setminus \{x_1\}) + A_r$, and that $\sum_{i=1}^{l-2} A'_i + (A'_{l-1} \setminus \{x_1\}) + A_r$ is H_a-periodic. Hence, since in view of Proposition 3.1 the compliment of puncture periodic set is aperiodic, it follows that (13) cannot hold for $\sum_{i=1}^{l-2} A'_i + (A'_{l-1} \setminus \{x_1\}) + A_r$, whence the induction is complete by letting $A'_{l-1} = A'_{l-1} \setminus \{x_1\}$ and letting $A'_l = A_r$. So we may assume that $|\sum_{i=1}^{l-1} A'_i + A_r| \neq |\sum_{i=1}^{l-1} A'_i| + |A_r| - 1$ for all $r > l-1$.

Hence, in view of the conclusion of the third paragraph of Case 1b, it follows that every set A_r with $r > l-1$ satisfies

$$|\sum_{i=1}^{l-1} A'_i + A_r| > |\sum_{i=1}^{l-1} A'_i| + |A_r| - 1. \tag{18}$$

Let $B_1, \ldots, B_{l'}$ be a nonempty subsequence of $A_l, \ldots, A_n$. If

$$|\sum_{i=1}^{l-1} A'_i + \sum_{i=1}^{l'} B_i| \leq |\sum_{i=1}^{l-1} A'_i| + \sum_{i=1}^{l'} |B_i| - (l'+1) + 1, \tag{19}$$

then, in view of (18) and Theorem 2.2, it follows that there exists a set B_w such that $\sum_{i=1}^{l-1} A'_i + \sum_{\substack{i=1 \\ i \neq w}}^{l'} B_i + (B_w \setminus \{x\}) = \sum_{i=1}^{l-1} A'_i + \sum_{i=l}^{l'} B_i$, for every $x \in B_w$. Hence from (18) and Theorem 3.1 it follows that an $x \in B_w$ can be found so that the induction is complete by letting $A'_l = B_w \setminus \{x\}$. So we may assume for any l' that (19) does not hold. Hence, since $|A_l| \geq 3$, then in view of (18), it follows that the induction is complete by applying Theorem 3.2 with $A = \sum_{i=l}^{l-1} A'_i$, $B = A_l$, and $C_i = A_{l+i}$.

Case 2: If $s \neq 2$, then suppose $k_1 \neq 1$, and if $s = 2$, then suppose $k_1 \neq 1$ or $k_2 \neq n$. Let s' be the minimal cardinality of a set A_i. Note from the assumptions of the case that $s' \leq 2$. Let $k \leq n$ be the index such that $|A_i| \geq s' + 2$ for $i \geq k$. Let $A_{j'}$ be a subset with $|A_{j'}| = s'$. Note, for $j \geq k$ and for every $t \in A_j \setminus A_{j'}$, that we can remove t from A_j and place t in $A_{j'}$ to form a new set $A'_{j'}$ with $|A'_{j'}| > |A_{j'}|$. Hence

$$\eta_t(\sum_{i=1}^{l} A_{b_i}, A_j) \geq 1, \tag{20}$$

where $A' = (A_{b_1}, \ldots, A_{b_l})$ is any nonempty subsequence of $A = (A_1, \ldots, A_n)$ that does not include the term A_j, since otherwise

$$|\sum_{\substack{i=1 \\ i \neq j,j'}}^{n} A_i + (A_{j'} \cup \{t\}) + (A_j \setminus \{t\})| \geq \sum_{i=1}^{n} |A_i| - n + 1, \tag{21}$$

contradicting the extremal assumptions originally assumed for A. From (20) and Theorem 2.2 it follows that

$$|\sum_{i=1}^{l} A_{b_i} + (A_j \setminus A'_j)| \geq |\sum_{i=1}^{l} A_{b_i}| + |(A_j \setminus A'_j)| - 1, \tag{22}$$

where $A' = (A_{b_1}, \ldots, A_{b_l})$ is any nonempty subsequence of $A = (A_1, \ldots, A_n)$ that does not include the term A_j, and A'_j is a proper subset of $A_j \setminus A_{j'}$.

Suppose that

$$|\sum_{i=1}^{l} A_{b_i}| \geq \sum_{i=1}^{l} |A_{b_i}| - l + 1, \tag{23}$$

for every nonempty subsequence $A' = (A_{b_1}, \ldots, A_{b_l})$ of $A = (A_1, \ldots, A_n)$. Since $|A_j| - |A_{j'}| \geq 2$, then in view of (23) and (22) with $A'_j = \{t\}$ and $A' = A \setminus (A_j)$, it follows that (21) holds, a contradiction to the extremal assumptions originally assumed for A, unless equality holds in (23) and (22) with $A'_j = \{t\}$ and $A' = A \setminus (A_j)$, and

$$|\sum_{\substack{i=1 \\ i \neq j}}^{n} A_i + (A_j \setminus \{t\})| = \sum_{i=1}^{n} |A_i| - n, \tag{24}$$

for each $t \in A_j \setminus A_{j'}$. However, since (21) cannot hold, then in view of Kneser's Theorem and (24), it follows that $\sum_{\substack{i=1 \\ i \neq j}}^{n} A_i + (A_j \setminus \{t\}) = \sum_{\substack{i=1 \\ i \neq j,j'}}^{n} A_i + (A_j \setminus \{t\}) + (A_{j'} \cup \{t\})$ is maximally H_{a_t}-periodic. Hence, in view of (20) with $A' = A \setminus (A_j)$ it follows that each $t \in A_j \setminus A_{j'}$ is the only element from its H_{a_t}-coset in A_j.

Suppose $A_{j'}$ does not contain an element from the same H_{a_t}-coset as t. Thus t is the unique element from its H_{a_t}-coset in $A_{j'} \cup \{t\}$. Hence, since $\sum_{\substack{i=1 \\ i \neq j}}^{n} A_i + (A_j \setminus \{t\}) = \sum_{\substack{i=1 \\ i \neq j,j'}}^{n} A_i + (A_j \setminus \{t\}) + (A_{j'} \cup \{t\})$ is maximally H_{a_t}-periodic, and in view of Kneser's Theorem, it follows that $|\sum_{\substack{i=1 \\ i \neq j',j}}^{n} A_i + (A_j \setminus \{t\}) + (A_{j'} \cup \{t\})| \geq |\sum_{\substack{i=1 \\ i \neq j',j}}^{n} A_i + (A_j \setminus \{t\})| + |(A_{j'} \cup \{t\})| - 1$. Hence from (23) and (22) with $A'_j = \{t\}$ and $A' = A \setminus (A_{j'}, A_j)$, it follows that (21) holds, a contradiction. So we may assume $\phi_{a_t}(t) \in \phi_{a_t}(A_{j'})$. Thus, since each $t \in A_j \setminus A_{j'}$ is the only element from its H_{a_t}-coset in A_j (from second paragraph of Case 2), it follows that $A_{j'} \nsubseteq A_j$. Hence $|A_j \setminus A_{j'}| \geq 3$.

Hence in view of (24), (20), (23) and (22) with $A'_j = \{t_1, t_2\}$ and $A' = A \setminus (A_j)$, it follows that

$$|\sum_{\substack{i=1\\i\neq j}}^{n} A_i + (A_j \setminus \{t_1, t_2\})| = \sum_{i=1}^{n} |A_i| - n - 1, \tag{25}$$

for any pair of distinct $t_1, t_2 \in A_j \setminus A_{j'}$. Hence, in view of (24) and (20) with $A' = A \setminus (A_j)$, it follows that $\eta_t(\sum_{\substack{i=1\\i\neq j}}^{n} A_i, A_j) = 1$ for each $t \in A_j \setminus A_{j'}$.

Since $\sum_{\substack{i=1\\i\neq j}}^{n} A_i + (A_j \setminus \{t\})$ is periodic, it follows that $\sum_{i=1}^{n} A_i$ is the disjoint union of that periodic set, say T, and all those elements of $\sum_{i=1}^{n} A_i$ that have precisely one representation in the sumset $\sum_{\substack{i=1\\i\neq j}}^{n} A_i + A_j$ and with that one representation using the term t. Since $\eta_t(\sum_{\substack{i=1\\i\neq j}}^{n} A_i, A_j) = 1$, it follows that there is precisely one such element of $\sum_{i=1}^{n} A_i$, say x, that has precisely one representation in the sumset $\sum_{\substack{i=1\\i\neq j}}^{n} A_i + A_j$ and with that one representation using the term t. Hence $\sum_{i=1}^{n} A_i = T \cup \{x\}$ is a reduced quasi-periodic decomposition of $\sum_{i=1}^{n} A_i$. Any periodic set has a reduced quasi-periodic decomposition with the aperiodic part empty, so by the characterization of reduced quasi-periodic decompositions given by Proposition 3.1, it follows that $\sum_{i=1}^{n} A_i$ cannot both have the reduced quasi-periodic decomposition $T \cup \{x\}$ as well as a reduced quasi-periodic decomposition with aperiodic part empty. Thus $\sum_{i=1}^{n} A_i$ must be aperiodic.

Next apply the Kneser Lemma with $C_0 = \sum_{i=1}^{n} A_i$, $C_1 = \sum_{\substack{i=1\\i\neq j}}^{n} A_i + (A_j \setminus \{t_1\})$, and $C_2 = \sum_{\substack{i=1\\i\neq j}}^{n} A_i + (A_j \setminus \{t_2\})$, where t_1 and t_2 are an arbitrary pair of distinct elements from $A_j \setminus A_{j'}$. Since $C_0 = \sum_{i=1}^{n} A_i$ is aperiodic (from the previous paragraph), it follows that $|H_{k_0}| = 1$ in the Lemma. Also note by their definitions that $H_{a_{t_1}} = H_{k_1}$ and $H_{a_{t_2}} = H_{k_2}$, in the notation of the Lemma. Since $\eta_t(\sum_{\substack{i=1\\i\neq j}}^{n} A_i, A_j) = 1$ for each $t \in A_j \setminus A_{j'}$, including t_1 and t_2, then it follows that $|C_1| = |C_2| = |C_0| - 1$. Hence the inequality given by the Kneser Lemma implies that either $|H_{k_1}| \leq 2$ or $|H_{k_2}| \leq 2$. Hence, since both H_{k_1} and H_{k_2} are nontrivial by their definition, it follows that either $|H_{k_1}| = 2$ or $|H_{k_2}| = 2$. If there were two distinct elements t_1 and t_2 from $A_j \setminus A_{j'}$ both with $|H_{k_1}| \neq 2$ and $|H_{k_2}| \neq 2$, then applying the above argument with these two t_i would yield a contradiction. Thus we can assume that $|H_{a_t}| = 2$ for all but at most one (say

t_0) $t \in A_j \setminus A_{j'}$.

Let $t \in A_j \setminus A_{j'}$ with $t \neq t_0$. Since $\sum_{i=1}^{n} A_i$ is aperiodic, it follows that every set A_i is aperiodic. Since $|H_{a_t}| = 2$, and since $\sum_{\substack{i=1\\i\neq j}}^{n} A_i + A_j \setminus \{t\}$ is maximally H_{a_t}-periodic, then from the remarks below the statement of Kneser's Theorem it follows that

$$|\sum_{\substack{i=1\\i\neq j}}^{n} A_i + (A_j \setminus \{t\})| = \sum_{\substack{i=1\\i\neq j}}^{n} |A_i| + |A_j \setminus \{t\}| - (n-1)|H_{a_t}| + \rho = \sum_{i=1}^{n} |A_i| - 2n + 1 + \rho,$$

where ρ is the number of H_{a_t}-holes contained collectively from the sets A_i, $i \neq j$, and from $A_j \setminus \{t\}$. Since each set A_i is aperiodic, it follows that each set A_i, $i \neq j$, contains at least one H_{a_t}-hole, and thus $\rho \geq n-1$, implying $|\sum_{\substack{i=1\\i\neq j}}^{n} A_i + A_j \setminus \{t\}| \geq \sum_{i=1}^{n} |A_i| - 2n + 1 + (n-1) = \sum_{i=1}^{n} |A_i| - n$. However, by (24) we know that equality holds in this inequality, and consequently it follows that each set A_i, $i \neq j$, must contain exactly one H_{a_t}-hole, and that $A_j \setminus \{t\}$ must contain no H_{a_t}-holes. Hence each set A_i is a union of an H_{a_t}-periodic set, say T, and a disjoint element, say x. However, since $|H_{a_t}| = 2$, then adding the other element (besides x) from the H_{a_t}-coset that contains x to the set A_i will complete the coset and make the resulting set H_{a_t}-periodic. Thus each A_i is a punctured H_{a_t}-periodic set. Hence, since $\phi_{a_t}(t) \in \phi_{a_t}(A_{j'})$ (from third paragraph of Case 2), and since $t \notin A_{j'}$, it follows that $A_{j'} \cup \{t\}$ is H_{a_t}-periodic, and that if $t' \in A_j \setminus A_{j'}$, $t' \neq t$, then $\phi_{a_t}(t') \notin \phi_{a_t}(A_{j'})$.

Since every set A_i is a punctured H_{a_t}-periodic set, and since $|H_{a_t}| = 2$, it follows that $|A_i|$ is odd for every $i \leq n$. Hence, since $s' \leq 2$, it follows that $s' = 1$, and that there is no set A_i with $|A_i| = s' + 1 = 2$.

Suppose

$$|\sum_{\substack{i=1\\i\neq j,j'}}^{n} A_i + (A_j \setminus \{t,t'\}) + (A_{j'} \cup \{t,t'\})| \leq \sum_{i=1}^{n} |A_i| - n, \tag{26}$$

for distinct $t, t' \in A_j \setminus A_{j'}$, $t \neq t_0$. Hence from Kneser's Theorem, it follows that $\sum_{\substack{i=1\\i\neq j,j'}}^{n} A_i + (A_j \setminus \{t,t'\}) + (A_{j'} \cup \{t,t'\})$ is maximally $H_{a'}$-periodic.

Suppose the inequality in (26) is strict. Hence, since

$$\sum_{\substack{i=1\\i\neq j}}^{n} A_i + (A_j \setminus \{t,t'\}) \subseteq \sum_{\substack{i=1\\i\neq j,j'}}^{n} A_i + (A_j \setminus \{t,t'\}) + (A_{j'} \cup \{t,t'\}),$$

it follows in view of (25) that

$$\sum_{\substack{i=1\\i\neq j,j'}}^{n} A_i + (A_j \setminus \{t,t'\}) + (A_{j'} \cup \{t,t'\}) = \sum_{\substack{i=1\\i\neq j}}^{n} A_i + (A_j \setminus \{t,t'\}).$$

Hence, in view of (25) and (24), it follows that $\sum_{\substack{i=1\\i\neq j,j'}}^{n} A_i+(A_j\setminus\{t,t'\})+(A_{j'}\cup\{t,t'\})$ is a punctured H_{a_t}-periodic set. Thus from Proposition 3.1 it follows that $\sum_{\substack{i=1\\i\neq j,j'}}^{n} A_i+(A_j\setminus\{t,t'\})+(A_{j'}\cup\{t,t'\})$ cannot be periodic, contradicting that $\sum_{\substack{i=1\\i\neq j,j'}}^{n} A_i+(A_j\setminus\{t,t'\})+(A_{j'}\cup\{t,t'\})$ is $H_{a'}$-periodic. So we may assume that equality holds in (26).

Since $\sum_{\substack{i=1\\i\neq j}}^{n} A_i+(A_j\setminus\{t,t'\})\subseteq \sum_{\substack{i=1\\i\neq j,j'}}^{n} A_i+(A_j\setminus\{t,t'\})+(A_{j'}\cup\{t\})$, then in view of (25) it follows that $|\sum_{\substack{i=1\\i\neq j,j'}}^{n} A_i+(A_j\setminus\{t,t'\})+(A_{j'}\cup\{t\})|\geq\sum_{i=1}^{n}|A_i|-n-1$. Suppose

$$|\sum_{\substack{i=1\\i\neq j,j'}}^{n} A_i+(A_j\setminus\{t,t'\})+(A_{j'}\cup\{t\})|>\sum_{i=1}^{n}|A_i|-n-1.$$

Hence, since $\sum_{\substack{i=1\\i\neq j,j'}}^{n} A_i+(A_j\setminus\{t,t'\})+(A_{j'}\cup\{t\})\subseteq\sum_{\substack{i=1\\i\neq j,j'}}^{n} A_i+(A_j\setminus\{t\})+(A_{j'}\cup\{t\})$, it follows in view of (24) that

$$\sum_{\substack{i=1\\i\neq j,j'}}^{n} A_i+(A_j\setminus\{t,t'\})+(A_{j'}\cup\{t\})=\sum_{\substack{i=1\\i\neq j,j'}}^{n} A_i+(A_j\setminus\{t\})+(A_{j'}\cup\{t\})=\sum_{\substack{i=1\\i\neq j}}^{n} A_i+(A_j\setminus\{t\}).$$

Hence in view of (26) it follows that

$$\sum_{\substack{i=1\\i\neq j,j'}}^{n} A_i+(A_j\setminus\{t,t'\})+(A_{j'}\cup\{t\})=\sum_{\substack{i=1\\i\neq j,j'}}^{n} A_i+(A_j\setminus\{t,t'\})+(A_{j'}\cup\{t,t'\})$$

is maximally H_{a_t}-periodic. Hence, since $\phi_{a_t}(t')\notin\phi_{a_t}(A_{j'})$ (from seventh paragraph of Case 2), since t is the only element from its H_{a_t}-coset in A_j (from second paragraph of Case 2), since $|H_{a_t}|=2$, and since each A_i is a punctured H_{a_t}-coset (from seventh paragraph of Case 2), it follows from Kneser's Theorem (by counting holes) that $|\sum_{\substack{i=1\\i\neq j,j'}}^{n} A_i+(A_j\setminus\{t,t'\})+(A_{j'}\cup\{t,t'\})|\geq\sum_{i=1}^{n}|A_i|-n+2$, contradicting (26). So we may assume that $|\sum_{\substack{i=1\\i\neq j,j'}}^{n} A_i+(A_j\setminus\{t,t'\})+(A_{j'}\cup\{t\})|=\sum_{i=1}^{n}|A_i|-n-1$.

Hence, since equality holds in (26), it follows that $\sum_{\substack{i=1\\i\neq j,j'}}^{n} A_i+(A_j\setminus\{t,t'\})+(A_{j'}\cup\{t\})$ is punctured from the $H_{a'}$-periodic set $\sum_{\substack{i=1\\i\neq j,j'}}^{n} A_i+(A_j\setminus\{t,t'\})+(A_{j'}\cup\{t,t'\})$, and thus is aperiodic by Proposition 3.1. However, since $A_{j'}\cup\{t\}$ is H_{a_t}-periodic (from seventh paragraph of Case

2), it follows that $\sum_{\substack{i=1\\i\neq j,j'}}^{n} A_i+(A_j\setminus\{t,t'\})+(A_{j'}\cup\{t\})$ is periodic, a contradiction. So we may assume (26) does not hold, i.e. that

$$|\sum_{\substack{i=1\\i\neq j,j'}}^{n} A_i+(A_j\setminus\{t,t'\})+(A_{j'}\cup\{t,t'\})|\geq\sum_{i=1}^{n}|A_i|-n+1, \tag{27}$$

for distinct $t,t'\in A_j\setminus A_{j'}$, $t\neq t_0$.

If $|A_j|-|A_{j'}|>2$, then in view of (27) it follows that the set partition obtained by moving t and t' from A_j to $A_{j'}$ satisfies (1) and contains one less set of cardinality s', contradicting the extremal conditions originally assumed for A. Thus we may assume $|A_j|-|A_{j'}|=2$. Hence $|A_j|=s'+2=3$. Consequently, since A_j and $A_{j'}$ with $|A_j|\geq s'+2$ and $|A_{j'}|=s'$ were arbitrary, and since there are no sets A_i with $|A_i|=s'+1$ (from eighth paragraph of Case 2), it follows that $|A_i|=1$ for $i<k$ and that $|A_i|=3$ for $i\geq k$. Thus $s=2$ and hence applying Case 1 to the $(n-k+1)$-set partition $A_k,A_{k+1},\ldots,A_n$ completes the proof. So we may assume (23) does not hold.

Since (23) does not hold, then let l be the minimal integer such that, allowing re-indexing,

$$|\sum_{i=1}^{l}A_i|<\sum_{i=1}^{l}|A_i|-l+1. \tag{28}$$

Hence from Kneser's Theorem it follows that $\sum_{i=1}^{l}A_i$ is maximally H_a-periodic. Since $s'\leq 2$, then in view of (20), Theorem 2.2 and the minimality of l, it follows that $|A_i|\leq s'+1\leq 3$ for $i\leq l$. Hence, in view of Kneser's Theorem and the minimality of l, it follows (by counting holes) that each A_i with $i\leq l$ is contained in an H_a-coset. Thus, since $\sum_{i=1}^{l}A_i$ is H_a-periodic, it follows that $\sum_{i=1}^{l}A_i$ is an H_a-coset. Let b, s_1, and s_2 be as defined in Case 1a. Since $\sum_{i=1}^{l}A_i$ is an H_a-coset, then in view of Proposition 2.2(ii), it follows that we can remove elements from the sets in A_i with $i\leq l$, yielding new, nonempty sets A_i', such that $s_1'\overset{def}{=}\sum_{i=1}^{l}|A_1'|\leq|H_a|+l-1$ and $\sum_{i=1}^{l}A_1'=\sum_{i=1}^{l}A_1$.

Let S' be a minimal length subsequence of the terms of S partitioned by the A_i where $i\geq l+1$, with an $(n-l)$-set partition, $B'=B_1,\ldots,B_{n-l}$, such that $|\sum_{i=1}^{n-l}\phi_a(B_i)|\geq b+1$ (since $\sum_{i=1}^{l}A_i'$ is an H_a-coset, such a subsequence exists by (1) and (5)). In view of Proposition 2.2(ii) it follows that $|S'|\leq(n-l)+b$.

Letting $s_2'=|S'|$, letting $r'=r$ for $s\geq 3$, and letting $r'=n-1$ for $s=2$, observe that the proof will be complete unless

$$s_2'+s_1'\geq(s-1)n+r'+2. \tag{29}$$

Hence from the conclusions of the last two paragraphs, it follows that

$$(s-1)n + r' + 2 \leq |H_a| + l - 1 + (n-l) + b,$$

implying $(s-1)n \leq \frac{s-1}{s-2}(|H_a|+b-r'-3) \leq 2(|H_a|+b-r'-3)$ for $s \geq 3$, and that $n \leq |H_a|+b-2$ for $s = 2$. Hence in view of (5), it follows that $b|H_a| \leq 2|H_a|+2b-5$, implying $(b-2)|H_a| \leq 2b-5$, whence $b \leq 1$. Since $|A_i| \leq s'+1 \leq 3$ for $i \leq l$, it follows from the minimality of l that $|A_i| = 2$ or $|A_i| = 3$ for all $i \leq l$. Hence, in view of (28), it follows that applying proposition 2.2(ii) to the A_i with $i \leq l$, yields sets $A'_i \subseteq A_i$ such that $\sum_{i=1}^{l} A'_i = \sum_{i=1}^{l} A_i$, such that $|\sum_{i=1}^{l} A'_i| = \sum_{i=1}^{l} |A'_i| - l + 1$, such that $|A'_r| \leq 2$ for some r, and such that the conditions of Case 1 hold for the subsequence of the A'_i consisting of those A'_i with $|A'_i| > 1$. Hence, since $|A'_r| \leq 2$ for some r, then applying Case 1 it follows that we may assume that $s'_1 \leq 2l$. Hence, since $b \leq 1$, and since $s'_2 \leq (n-l)+b$, it follows that $s'_1 + s'_2 \leq n + l + 1$. Thus from (29) it follows that $n + l + 1 \geq 2n + 1$, whence $n \leq l$ contradicting (1) or (28), and completing the proof. □

4 Mostly Monochromatic Zero-Sums

Given $\alpha \in \mathbb{Z}_m$, let $\overline{\alpha}$ be the least positive integer representative of α. The proof of Theorem 1.4, which we begin below, follows a method introduced by Gao and Hamidoune [10].

Proof Theorem 1.4. Let $|\{a\} \cap S| = n_0$, let $|\{b\} \cap S| = n_1$, and let $t = |S| - n_0 - n_1$. We may w.l.o.g. assume $|S| = 2m-1$, $n_1 \leq n_0 \leq m-1$, and $a = 0$. Hence, since by hypothesis

$$t \leq \left\lfloor \frac{m}{2} \right\rfloor, \tag{30}$$

it follows that

$$\left\lceil \frac{m}{2} \right\rceil \leq m - t \leq n_1 \leq n_0 \leq m-1, \tag{31}$$

and, in view of the pigeonhole principle, that

$$m - \left\lfloor \frac{t+1}{2} \right\rfloor \leq n_0. \tag{32}$$

Let c be the order of b. Suppose first that $c < m$. Let l be the least integer such that $\lfloor \frac{t+1}{2} \rfloor \leq l$ and $c|l$. Observe $l \leq \lfloor \frac{t+1}{2} \rfloor + c - 1$. Hence, if $c < \frac{m}{3}$, then in view of (30) it follows that $l \leq \lfloor \frac{m+2}{4} \rfloor + \frac{m}{4} - 1 \leq \lceil \frac{m}{2} \rceil$. On the other hand, if $c \geq \frac{m}{3}$, then from (30) it follows that $\lfloor \frac{t+1}{2} \rfloor \leq c$, whence $l = c \leq \lceil \frac{m}{2} \rceil$. Hence, in view of (31) and (32), it follows in both cases that the proof is complete by selecting l terms equal to b and $m-l$ terms equal to 0. So we may assume that $c = m$, whence G is cyclic and w.l.o.g. $b = 1$.

Let $W = w_1, w_2, \ldots, w_l$, be a subsequence of the terms of S not equal to 0 or 1, and let $\sum_{i=1}^{l} w_i = w$. Observe that the m-term sequence

$$(\underbrace{0,\ldots,0}_{\overline{w}-l},\underbrace{1,\ldots,1}_{m-\overline{w}},w_1,\ldots,w_l)$$

is zero-sum provided $\overline{w} \geq l$. Hence, in view of (31), it follows that if $\overline{w} \geq \lfloor \frac{m}{2} \rfloor + l$, then

$$\overline{w} \geq n_0 + l + 1, \tag{33}$$

and if $l \leq \overline{w} \leq \lceil \frac{m}{2} \rceil$, then

$$\overline{w} \leq m - n_1 - 1, \tag{34}$$

else the proof is complete.

Let $Y = y_1, \ldots, y_{r_y}$ be the subsequence of S consisting of terms y_i such that $1 < \overline{y_i} \leq \frac{m}{2}$, and let $Z = z_1, \ldots, z_{r_z}$ be the subsequence of S consisting of terms z_i such that $\frac{m}{2} < \overline{z_i} \leq m-1$. Applying (33) with $W = \{z_i\}$, it follows that $\overline{z_i} \geq n_0 + 2$ for all i. Hence, since $\frac{m}{2} < \overline{z_i} \leq m-1$, then in view of (30), (32), and (33) applied to $W = z_1, \ldots, z_{l-1}$, it follows from an easy inductive argument passing from $l-1$ to l that $\lfloor \frac{m}{2} \rfloor + l \leq \overline{\sum_{i=1}^{l} z_i}$ for all $l \in \{1, \ldots, r_z\}$. Hence, since $\frac{m}{2} < \overline{z_i} \leq m-1$, it follows that $\overline{\sum_{i=1}^{l} z_i} \leq m - l$. Consequently from (33) applied with $W = Z$, it follows that

$$r_z \leq \frac{m - n_0 - 1}{2}. \tag{35}$$

Let $Y' = y'_1, \ldots, y'_l$ be a subsequence of Y with length l. We next show by induction on l, passing from $l-1$ to l, that

$$\overline{\sum_{i=1}^{l} y'_i} \leq \left\lfloor \frac{m}{2} \right\rfloor + l - 1, \tag{36}$$

for all $l \in \{1, \ldots, r_y\}$. The case $l = 1$ follows from the definition of Y. Since $2m-1 = n_0+n_1+t$, then applying (34) with $W = \{y_i\}$, it follows that $\overline{y'_i} \leq t - m + n_0$ for all i. Hence by induction hypothesis it follows that

$$n_0 - \left\lceil \frac{m}{2} \right\rceil + l - 2 + t \geq \overline{\sum_{i=1}^{l} y'_i}. \tag{37}$$

If (36) does not hold, then applying (33) with $W = Y'$, it follows that $\overline{\sum_{i=1}^{l} y'_i} \geq n_0 + l + 1$. Hence from (37) it follow that $t \geq \lceil \frac{m}{2} \rceil + 3$, contradicting (30). So we may assume that (36) holds.

We proceed to show that

$$\overline{\sum_{i=1}^{l} y'_i} = \sum_{i=1}^{l} \overline{y'_i}. \tag{38}$$

Since $\overline{y_i'} \leq \frac{m}{2}$, it follows that (38) holds for $l = 1$ and $l = 2$. Assume inductively that (38) holds up to $(l-1)$, where $l \geq 3$. Letting j, $j' \in \{1, \ldots, l\}$ be arbitrary distinct indices, it follows in view of (36) and the induction hypothesis that $\sum_{\substack{i=1\\i\neq j}}^{l} \overline{y_i'} = \overline{\sum_{\substack{i=1\\i\neq j}}^{l} y_i'} \leq \lfloor \frac{m}{2} \rfloor + l - 2$. Hence, using the estimate $\overline{y_i'} \geq 2$ for $i \neq j'$, it follows that

$$\overline{y_{j'}'} \leq \left\lfloor \frac{m}{2} \right\rfloor - l + 2, \tag{39}$$

for all $j' \in \{1, \ldots, l\}$. But then from (39), induction hypothesis and (36), it follows that

$$\sum_{i=1}^{l} \overline{y_i'} = \overline{y_l'} + \sum_{i=1}^{l-1} \overline{y_i'} = \overline{y_l'} + \overline{\sum_{i=1}^{l-1} y_i'} \leq \left\lfloor \frac{m}{2} \right\rfloor - l + 2 + \left\lfloor \frac{m}{2} \right\rfloor + l - 2 = 2\left\lfloor \frac{m}{2} \right\rfloor \leq m,$$

from which (38) immediately follows.

In view of (35) and (32), it follows that

$$r_y \geq \frac{3t+1}{4}. \tag{40}$$

Let l be the maximal integer for which there exists a subsequence $Y' = y_1', \ldots, y_l'$ of Y satisfying $\sum_{i=1}^{l} \overline{y_i'} \leq \lceil \frac{m}{2} \rceil$. Hence, since $2m - 1 = n_0 + n_1 + t$, and since $\overline{y_i} \geq 2$, it follows, in view of (34) and (38), that

$$2l \leq \sum_{i=1}^{l} \overline{y_i'} \leq n_0 + t - m. \tag{41}$$

Hence, since $m - n_0 \geq 1$, it follows that $l \leq \frac{t-1}{2}$. Hence from (40) it follows that there are at least $\lceil \frac{t+3}{4} \rceil$ terms of Y not in the maximal subsequence Y'. Furthermore, since $l \geq 1$, it follows that $t \geq 3$. Let $A = a_1, \ldots, a_{\lceil (t+3)/4 \rceil}$ be a subsequence of $Y \setminus Y'$. Define α by $\sum_{i=1}^{l} \overline{y_i'} = n_0 + t - m - \alpha$. From (41) it follows that $\alpha \geq 0$. Hence, in view of the maximality of Y', it follows that $\overline{y} \geq \lceil \frac{m}{2} \rceil + m - n_0 - t + 1 + \alpha$ for each $y \in Y \setminus Y'$. Hence by considering lower and upper bounds for $\sum\limits_{a \in A} \overline{a} + \sum\limits_{y' \in Y} \overline{y'}$, it follows, in view of (36) and (38), that

$$\left\lceil \frac{t+3}{4} \right\rceil \left(\left\lceil \frac{m}{2} \right\rceil + m - n_0 - t + 1 + \alpha \right) + (n_0 + t - m - \alpha) \leq \left\lfloor \frac{m}{2} \right\rfloor + l + \left\lceil \frac{t+3}{4} \right\rceil - 1.$$

Hence, since $\alpha \geq 0$, since $t \geq 3$, since $m - n_0 \geq 1$, and in view of (30), it follows that if m is odd, or $m - n_0 \geq 2$, or $t < \lfloor \frac{m}{2} \rfloor$, then the above inequality implies $l \geq \frac{t+1}{2}$, a contradiction to $l \leq \frac{t-1}{2}$. Hence, in view of (30), we may assume m is even, $t = \frac{m}{2}$, and $n_0 = m - 1$. Hence from (35) it follows that $r_y = \frac{m}{2}$. Thus from (38) it follows that $y_i = 2$ for all i, whence in view of (31) the proof is complete by selecting $\frac{m}{2}$ terms equal to 0 and $\frac{m}{2}$ terms equal to 2. $\square$

5 Applications to Erdős-Ginzburg-Ziv

We begin this section first with the following simple proposition, which is easily proved by induction on s.

Proposition 5.1. *Let m and s be positive integers, and let S be a sequence of elements from a finite group of order m. If $|S| \geq m + 2s - 1$, then there exist two disjoint s-term subsequences of S whose sums are equal.*

As a simple corollary to Theorems 2.1, 1.3 and 1.4, we are now ready extend the Erdős-Ginzburg-Ziv Theorem to a class of hypergraphs.

Theorem 5.1. *Let $\mathcal{H}$ be a finite m-uniform hypergraph, let $e \in E(\mathcal{H})$, and let $\mathcal{H}'$ be the subhypergraph obtained by removing the edge e and all monovalent vertices contained in e. If $f_{zs}(\mathcal{H}') \leq 2|V(\mathcal{H}')| - 1$ and e has at least $\lceil \frac{m}{2} \rceil$ monovalent vertices, then $f_{zs}(\mathcal{H}) \leq 2|V(\mathcal{H})| - 1$.*

Proof. Let S denote the sequence given by a coloring $\Delta : V \to \mathbb{Z}_m$, where $n = |V(\mathcal{H})|$ and $V = V(K^m_{2n-1})$. Let s be the number of non-monovalent vertices in e. Note that by assumption $s \leq \lfloor \frac{m}{2} \rfloor$. We may assume the multiplicity of each term in S is at most $n-1$, else there will be an edgewise zero-sum copy of $\mathcal{H}$ with all edges monochromatic. Hence, if there exists a subset $X \subseteq V$ such that $|X| \leq s - 2 \leq \lfloor \frac{m}{2} \rfloor - 2$ and $|\Delta(V \setminus X)| \leq 2$, then setting aside $n - m$ terms colored by a_i for each of the two $a_i \in \Delta(V \setminus X)$ and applying Theorem 1.4 to the remaining $2m - 1$ terms, it follows that there exists an edge-wise zero-sum copy of $\mathcal{H}$ with the vertices of e colored by the zero-sum sequence given by Theorem 1.4, and all other edges monochromatic. Otherwise, since $s \leq \lfloor \frac{m}{2} \rfloor$, it follows from Proposition 2.1 that there exists an $(2n-m)$-set partition P of S with at least $2n - 2m + s$ cardinality one sets. Since $s \leq \lfloor \frac{m}{2} \rfloor$, then applying Theorem 2.1 to P yields two cases.

If Theorem 2.1(i) holds, then let A be the set partition given by (i), and let A' be the $(m-s)$-set partition obtained by deleting $2n-2m+s$ cardinality one sets from A. Applying Theorem 1.3 to the set partition A' yields an $(m-s)$-set partition A'' that contains at most $2(m-s)$ terms of S, and whose sumset is $\mathbb{Z}_m$. This leaves at least $2n-1-2(m-s) = 2(n-m+s)-1 \geq 2|V(\mathcal{H}')|-1$ vertices not contained in any term of A''. Thus, since $f_{zs}(\mathcal{H}') \leq 2|V(\mathcal{H}')| - 1$, it follows that there exists an edge-wise zero-sum copy of $\mathcal{H}'$ not containing any vertices contained in A''. Hence, since the sumset of terms in the $(m-s)$-set partition A'' is $\mathbb{Z}_m$, it follows that we can find $m - s$ vertices from A'' which together with the vertices of $\mathcal{H}'$ form an edge-wise zero-sum copy of $\mathcal{H}$.

If Theorem 2.1(ii) holds, then there exists a proper nontrivial subgroup H_a of index a such that all but at most $a - 2$ terms of S are from the coset $\alpha + H_a$, and w.l.o.g. by translation we may assume $\alpha = 0$; furthermore, there exists a subsequence S' of S of length at most $2n-1-(a-2)$ with an $(2n-m)$-set partition $P' = P'_1, \ldots, P'_{2n-m}$ satisfying $\sum_{i=1}^{2n-m} P'_i = H_a$. Hence, since $\frac{m}{a} \leq m-s \leq 2n-m$, then by applying Proposition 2.2(i) followed by Proposition 2.2(ii), it follows that there exists a subsequence S'' of S' satisfying $|S''| \leq m-s+\frac{m}{a}-1$ and which has an $(m-s)$-set partition P'' the sumset of whose terms is H_a. Hence it follows that there are at least $2n-1-(m-s+\frac{m}{a}-1)-(a-2) \geq 2n-1-2(m-s)$ terms of S that are not used in the set partition P'', and which are from H_a, whence the proof is complete as it was in the previous paragraph. □

We can now prove our main results.

Proof Theorem 1.1. If $\mathcal{H}$ has one edge, this is precisely a restatement of the Erdős-Ginzburg-Ziv Theorem. Hence the upper bound for Theorem 1.1 follows from Theorem 5.1 and induction on the number of edges (relaxing the connectedness condition), while the lower bound for connected $\mathcal{H}$ is trivial. □

Proof Theorem 1.2. Let S denote the sequence given by a coloring $\Delta : V \to \mathbb{Z}_m$, where $n = |V(\mathcal{H})|$ and $V = V(K^m_{2n-1})$. Let the two edges of $\mathcal{H}$ be A and B. If $|A \cap B| < \lceil \frac{m}{2} \rceil$, then the proof is complete by Theorem 1.1. So we may assume $|A \cap B| \geq \lceil \frac{m}{2} \rceil$. Let $s = m - |A \cap B|$. Note $n = m + s$, $|S| = 2m + 2s - 1$, and $s \leq \lfloor \frac{m}{2} \rfloor$.

We may also assume the multiplicity of each term in S is at most $n-1$, else there will be an edgewise zero-sum copy of $\mathcal{H}$ with all edges monochromatic. Hence, if there exists a subset $X \subseteq V$ such that $|X| \leq \lceil \frac{m}{2} \rceil - 2$ and $|\Delta(V \setminus X)| \leq 2$, then setting aside s terms colored by a_i for each of the two $a_i \in \Delta(V \setminus X)$ and applying Theorem 1.4 to the remaining $2m-1$ terms, it follows that there exists an edge-wise zero-sum copy of $\mathcal{H}$ with the vertices of A colored by the zero-sum sequence given by Theorem 1.4, and with $V(\mathcal{H}) \setminus (A \cap B)$ monochromatic. Otherwise, it follows from Proposition 2.1 that there exists an $(m+2s)$-set partition P of S with at least $\lceil \frac{m}{2} \rceil + 2s$ cardinality one sets. Applying Theorem 2.1 to P yields two cases.

If Theorem 2.1(i) holds, then let A be the set partition given by (i), and let A' be the $\lfloor \frac{m}{2} \rfloor$-set partition obtained by deleting $\lceil \frac{m}{2} \rceil + 2s$ cardinality one sets from A. Applying Theorem 1.3 to the set partition A' yields an $\lfloor \frac{m}{2} \rfloor$-set partition A'' that contains at most m terms of S, and whose sumset is $\mathbb{Z}_m$. This leaves at least $m + 2s - 1$ vertices not contained in any term of A''. Hence from Proposition 5.1 it follows that there are two disjoint s-term subsequences S_1 and S_2, none of whose terms are contained in a term of A'', and whose sums are equal to (say) t. Since $s \leq \lfloor \frac{m}{2} \rfloor$, then let T be a subsequence of length $m - s - \lfloor \frac{m}{2} \rfloor$ whose terms are not contained in

S_1, S_2, nor any term of A''. Let t' be the sum of the terms in T, if T is nonempty, and otherwise let $t' = 0$. Since $s \le \lfloor \frac{m}{2} \rfloor$, and since the sumset of A'' is $\mathbb{Z}_m$, it follows that we may choose $\lfloor \frac{m}{2} \rfloor$ terms of S from A'' whose sum is $-(t+t')$, which along with S_1, S_2 and T yields an edgewise zero-sum copy of $\mathcal{H}$ with the terms from A'' and T contained in $A \cap B$.

If Theorem 2.1(ii) holds, then there exists a proper nontrivial subgroup H_a of index a such that all but at most $a-2$ terms of S are from the coset $\alpha + H_a$, and w.l.o.g. by translation we may assume $\alpha = 0$; furthermore, there exists a subsequence S' of S of length at most $2n-1-(a-2)$ with an $(m+2s)$-set partition $P' = P'_1, \ldots, P'_{m+2s}$ satisfying $\sum_{i=1}^{m+2s} P'_i = H_a$. Hence, since $\frac{m}{a} \le m-s \le m+2s$, then by applying Proposition 2.2(i) followed by Proposition 2.2(ii), it follows that there exists a subsequence S'' of S' satisfying $|S''| \le m - s + \frac{m}{a} - 1$ and which has an $(m-s)$-set partition P'' the sumset of whose terms is H_a. Hence it follows that there are at least $2m + 2s - 1 - (a-2) - (m - s + \frac{m}{a} - 1) = m + 3s - \frac{m}{a} - a + 2 \ge \frac{m}{a} + 2s - 1$ terms of S that are not used in the set partition P'', and which are from H_a, whence the proof is complete as it was in the previous paragraph. □

We conclude by giving an example of a fairly simple hypergraph on $(\lfloor \frac{m}{2} \rfloor + 3)(\lceil \frac{m}{2} \rceil - 1)$ vertices with every edge having at least $\lceil \frac{m}{2} \rceil - 2$ monovalent vertices, but which does not edgewise zero-sum generalize, showing that the $\lceil \frac{m}{2} \rceil$ bound given in Theorems 1.1 and 5.1 can be improved at best to $\lceil \frac{m}{2} \rceil - 1$. Let X be a set of $\lfloor \frac{m}{2} \rfloor + 3$ vertices, and for each $\lfloor \frac{m}{2} \rfloor + 2$ subset X' of X, define an edge of the hypergraph $\mathcal{H}$ to be X' along with $\lceil \frac{m}{2} \rceil - 2$ monovalent vertices disjoint from X. For the coloring of the complete graph, let Δ consist entirely of an equal number of vertices colored by 0 and 1, and one vertex colored by $\lceil \frac{m}{2} \rceil$. Hence, since the only non-monochromatic m-term zero-sum sequence is $(\underbrace{0, \ldots, 0}_{\lceil \frac{m}{2} \rceil - 1}, \underbrace{1, \ldots, 1}_{\lfloor \frac{m}{2} \rfloor}, \lceil \frac{m}{2} \rceil)$, it follows that any edgewise zero-sum copy of $\mathcal{H}$ must have $|\Delta(X)| = 3$, which, since there can be no non-monochromatic zero-sum edge using only the colors 0 and 1, is impossible.

As a final remark, we note that the arguments used in this section to obtain upper bounds for colorings with $\mathbb{Z}_m$ work equally well for colorings with any abelian group G of order m, although in the noncyclic case the matching lower bound constructions do not hold.

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