On the Intersection of two m-sets and the Erdős-Ginzburg-Ziv Theorem

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Abstract

We prove the following extension of the Erdős-Ginzburg-Ziv Theorem. Let m be a positive integer. For every sequence $\{a_i\}_{i\in I}$ of elements from the cyclic group $\mathbb{Z}_m$, where $|I| = 4m-5$ (where $|I| = 4m-3$), there exist two subsets A, $B \subseteq I$ such that $|A\cap B| = 2$ (such that $|A\cap B| = 1$), $|A| = |B| = m$, and $\sum\limits_{i\in A} a_i = \sum\limits_{i\in b} b_i = 0$.

1 Introduction

Since the seminal theorem of Erdős-Ginzburg-Ziv (EGZ) [13] [14] [1]—which states that any sequence of $2m-1$ elements from a finite abelian group of order m contains an m-term subsequence whose terms sum to zero—many generalizations, analogs, related problems [17] [15] [1] [2], and what are known as generalizations in the sense of EGZ for edge colorings of graphs [7] [16] as well as for colorings of the integers [12], were published. Two surveys appeared in [3] [5]. In the early 1990's, the first author posed the following related conjecture.

Conjecture 1.1. *Let m be a positive integer. For every sequence $\{a_i\}_{i\in I}$ of elements from the cyclic group $\mathbb{Z}_m$, where $|I| = 4m-5$ (where $|I| = 4m-3$), there exist two subsets A, $B \subseteq I$ such that $|A\cap B| = 2$ (such that $|A\cap B| = 1$), $|A| = |B| = m$, and $\sum\limits_{i\in A} a_i = \sum\limits_{i\in b} b_i = 0$.*

While the case $|A\cap B| = 1$ follows directly from the Cauchy-Davenport Theorem [6] for m prime, there were no tools to attack the case $|A\cap B| = 2$, until recently. The main tool

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to handle this kind of problem was developed by the second author [10]. It is stated below as Theorem 2.1. The aim of this note is to affirm the conjecture above. It is worthwhile to note that a continuation by the second author along similar lines will appear in [9].

2 Preliminaries

Let G denote an abelian group of order m, and let S be a sequence of elements from G. The length of S is denoted by $|S|$. If A, $B \subseteq G$, then their sumset, $A + B$, is the set of all possible pairwise sums, i.e. $\{a + b \mid a \in A, b \in B\}$. Furthermore, an n-set partition of S is a sequence of n nonempty subsequences of S, pairwise disjoint as sequences, such that every term of S belongs to exactly one subsequence, and the terms in each subsequence are distinct. Thus such subsequences can be considered sets. Let φ be the function which takes a sequence to its underlying set, so that if $S = (0, 0, 1, 2, 0, 2, 2)$, then $\varphi(S) = \{0, 1, 2\}$. For $\alpha \in \mathbb{Z}_m$, let $\overline{\alpha}$ denote the least positive integer representative of α. If S' is a subsequence of S, then $S \setminus S'$ denotes the subsequence of S obtained by deleting the terms of S' in S.

The following [10] [8] [11] is a recent composite analog of the Cauchy-Davenport Theorem [6].

Theorem 2.1. *Let S be a sequence of elements from an abelian group G of order m with an n-set partition $P = P_1, \ldots, P_n$, and let p be the smallest prime divisor of m. Then either:*

(i) there exists an n-set partition $A = A_1, A_2, \ldots, A_n$ of S such that:

$$|\sum_{i=1}^{n} A_i| \geq \min\{m,\ (n+1)p,\ |S| - n + 1\};$$

furthermore, if $n' \geq \frac{m}{p} - 1$ is an integer such that P has at least $n - n'$ cardinality one sets and if $|S| \geq n + \frac{m}{p} + p - 3$, then we may assume there are at least $n - n'$ cardinality one sets in A, or

(ii) (a) there exists $\alpha \in G$ and a nontrivial proper subgroup H_a of index a such that all but at most $a - 2$ terms of S are from the coset $\alpha + H_a$; and (b) there exists an n-set partition $A_1, A_2, \ldots, A_n$ of the subsequence of S consisting of terms from $\alpha + H_a$ such that $\sum_{i=1}^{n} A_i = n\alpha + H_a$.

When using the above theorem, the following basic proposition about n-set partitions is useful [2].

Proposition 2.1. *A sequence S has an n-set partition A if and only if the multiplicity of each element in S is at most n and $|S| \geq n$. Furthermore, a sequence S with an n-set partition* *has an n-set partition $A' = A_1, \ldots, A_n$ such that $||A_i| - |A_j|| \leq 1$ for all i and j satisfying $1 \leq i \leq j \leq n$.*

Finally, we need the following theorem which describes the extremal instances for EGZ [4].

Theorem 2.2. *Let S be a sequence of elements from $\mathbb{Z}_m$. If $|S| = 2m-2$ and S contains no m-term zero-sum subsequence, then S contains two distinct residues, whose difference is coprime to m, each with multiplicity $m-1$.*

3 The Proof

Let S be a sequence of elements from $\mathbb{Z}_m \stackrel{def}{=} G$ with $|S| = 4m-5$ (with $|S| = 4m-3$). If there exists $\alpha \in G$ such that $|\varphi^{-1}(\alpha)| \geq 2m-2$ (such that$|\varphi^{-1}(\alpha)| \geq 2m-1$), then the proof is complete with both m-term subsequences monochromatic. Hence we may assume $|\varphi(S)| \geq 3$, else the proof is complete by the pigeonhole principle.

Suppose there does not exist a subsequence S' of S with $|S'| = 2m-3$ (with $|S'| = 2m-2$), such that there exist an $(m-2)$-set partition P of S' (such that there exists an $(m-1)$-set partition of S'). Hence, since $|\varphi(S)| \geq 3$, it follows from Proposition 2.1 that there is $\alpha \in G$ with $|\varphi^{-1}(\alpha)| \geq 3m-3$ (with $|\varphi^{-1}(\alpha)| \geq 3m-1$), and the result follows from the arguments from the first paragraph. So we may assume such S' exists.

Since $|S \setminus S'| = 2m-2$ (since $|S \setminus S'| = 2m-1$), it follows from Theorem 2.2 that there is an m-term zero-sum subsequence of $S \setminus S'$, unless w.l.o.g. $\varphi(S \setminus S') = \{0, 1\}$, with both 0 and 1 occurring with multiplicity $m-1$ in $S \setminus S'$ (it follows from EGZ that there is an m-term zero-sum subsequence of $S \setminus S'$ regardless). We can avoid this case by swapping a 0 or 1 from $S \setminus S'$ with a term β from S' with $\beta \neq 1$ and $\beta \neq 0$, unless, up to order, $S = (\underbrace{0, 0, \ldots, 0}_{2m-3}, \underbrace{1, 1, \ldots, 1}_{2m-3}, \gamma)$, with $\gamma \neq 0$ and $\gamma \neq 1$; but it is easily checked, since $(\gamma, \underbrace{1, \ldots, 1}_{m-\overline{\gamma}}, \underbrace{0, \ldots, 0}_{\overline{\gamma}-1})$ is zero-sum, that the sequence $(\underbrace{0, 0, \ldots, 0}_{2m-3}, \underbrace{1, 1, \ldots, 1}_{2m-3}, \gamma)$ satisfies conjecture 1.1. So we may assume that there is a m-term zero-sum subsequence in $S \setminus S'$, say T.

Let $S'' = S \setminus T$, and let P' be a $(2m-4)$-set partition of S'' (let P' be a $(2m-2)$-set partition of S'') obtained by adding the terms of $(S \setminus S') \setminus T$ to P as singleton sets. Fix two elements in T, say $\{t_1, t_2\} = T'$ (fix an element in T, say $\{t_1\} = T'$). Applying Theorem 2.1 to P', it follows that either (i) holds and hence there exist $m-2$ elements from S'' (there exist $m-1$ elements from S'') which along with T' form a m-term zero-sum sequence, and the proof is complete, or else (ii) holds and hence, w.l.o.g. by translation, there exists a proper nontrivial subgroup $H \leq G$ with index a such that all but at most $a-2$ terms of S'' are from H. Note that this proves the theorem for m prime.

We proceed by induction on the number of primes in the factorization of m. Hence, since $4\frac{m}{a} - 5 \leq 3m - 3 - a$ (since $4\frac{m}{a} - 3 \leq 3m - 1 - a$), it follows by induction hypothesis that there are two $\frac{m}{a}$-term zero-sum subsequences of S'', A and B, that share exactly two terms (that share exactly one term). Thus, since $(2a-3)\frac{m}{a} + 2\frac{m}{a} - 1 \leq 3m - 3 - a - (2\frac{m}{a} - 2)$ (since $(2a-3)\frac{m}{a} + 2\frac{m}{a} - 1 \leq 3m - 1 - a - (2\frac{m}{a} - 1)$), it follows by $2a-2$ applications of the Erdős-Ginzburg-Ziv Theorem with the group H_a that there exist two m-term zero-sum subsequences A' and B', with A a subsequence of A', with B a subsequence of B', and with A' and B' sharing exactly two terms (sharing exactly one term), completing the proof. $\square$

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