On Four Color Monochromatic Sets with Nondecreasing Diameter

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December 11, 2008

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key words: nondecreasing diameter, Rado, Ramsey

Abstract

Let m and r be positive integers. Define $f(m, r)$ to be the least positive integer N such that for every coloring of the integers $1, \ldots, N$ with r colors there exist monochromatic subsets B_1 and B_2 (not necessarily of the same color), each having m elements, such that (a) $max(B_1) - min(B_1) \leq max(B_2) - min(B_2)$, and (b) $max(B_1) < min(B_2)$. We improve previous upper bounds to determine that $f(m, 4) = 12m - 9$.

*Research for this paper was done during the 2000 REU Program held at the University of Idaho under NSF grant DMS9820520, and served as partial fulfillment of the requirements for graduating with departmental honors at Bates College, Lewiston, ME.

1 Introduction

In recent years, progress has been made in the field of Generalized Ramsey Theory for colorings of the integers. Besides results related to Van der Waerden's Theorem [12], [7] and Rado's Dissertation [10], [7], exact Rado numbers have been determined for various equations. However, most of these results deal only with 2-colorings of the integers, e.g. [2], [9], [11]. Along different lines, N. Alon and J. Spencer [1], and T. C. Brown, P. Erős, and A. R. Freedman [5], considered configurations of a more strictly geometric nature. Bialostocki, Erdős, and Lefmann proposed another geometric Rado-type problem [3]: the determination of the function $f(m, r)$, described below.

Let m, r and k be positive integers. For finite subsets $X, Y \subseteq \mathbf{Z}$, the *diameter* of X, denoted by $diam(X)$, is defined as $max(X) - min(X)$. Moreover, we say that $X <_p Y$ if and only if $max(X) < min(Y)$. A set X, colored with the elements from the cyclic group $\mathbf{Z}_m$, is called *zero-sum* if the sum of the colors of all the elements of X is 0. Define $f(m, r)$ to be (define $f_{zs}(m, 2k + s)$, where $s = 0$ or 1, to be) the least integer N such that for every coloring of $[1, N] = \{1, \ldots, N\}$ with r colors (with the elements from the disjoint union of $i = 1, \ldots, k$ labeled copies of the cyclic group, $\mathbf{Z}_m^i$, of residues modulo m, denoted $\mathbf{Z}_m^{(k)}$, and if $s = 1$, by an additional color class $\infty \notin \mathbf{Z}_m^{(k)}$), there exist m-element subsets $B_1, B_2 \subseteq [1, N]$ such that: (a) B_i is monochromatic, for $i = 1, 2$ (B_i is either zero-sum in $\mathbf{Z}_m^j$ for some $j \in [1, k]$ or monochromatic in ∞, for $i = 1, 2$); (b) $B_1 <_p B_2$; and (c) $diam(B_1) \leq diam(B_2)$.

Their interest in the function $f(m, r)$ was related to a conjecture they posed, concerning a zero-sum generalization along the lines of the Erdős-Ginzburg-Ziv Theorem [7], that

$$f_{zs}(m,\, r) = f(m,\, r) \text{ for all } r \geq 2.$$

They were able to determine that $f_{zs}(m,\, 2) = f(m,\, 2) = 5m{-}3$, that $f_{zs}(m,\, 3) = f(m,\, 2) = 9m - 7$, and that $12m - 9 \leq f(m,\, 4) \leq 13m - 11$, as well as give general bounds. Bolobás, Erdős, and Jin [4] significantly improved these bounds in the monochromatic case when $m = 2$. In this paper, we show that $f(m,\, 4) = 12m - 9$. This result, in addition to a result from [6], shows that $f_{zs}(m,\, 4) = 12m - 9$ as well.

2 The Proof of $f(m,\, 4) = 12m - 9$

Let $\Delta : X \to C$ be a coloring of a finite set X by a set of colors C. For $C' \subseteq C$ and $Y \subseteq X$ we use the following notation: (a) $first_n(C')$ is the n-th smallest integer colored by an element from C'; (b) $last_n(C)$ is the n-th greatest integer colored by an element from C; (c) $first(C') = first_1(C')$; and (d) $last(C') = last_1(C')$. For the sake of simplicity, a coloring $\Delta : [1,\, N] \to C$ will be denoted by the string $\Delta(1)\Delta(2)\Delta(3)\ldots\Delta(N)$, and x^i will be used to denote the string $xx\ldots x$ of length i. Hence $\Delta : [1,\, 6] \to \{0,1\}$, where $\Delta[1,\, 2] = 0$, $\Delta(3) = 1$, and $\Delta[4, 6] = 0$, may be represented by the string $0^2 1 0^3$.

The following theorem of Erdős, Bialostocki, and Lefmann [3] was used in their proof of $f(m,\, 4)$ for the cases $r = 2$ and 3, and will be needed for the $r = 4$ case as well.

Theorem 1. *Let $m \geq 2$ be an integer, and let $\Delta : [1, 3m - 2] \to \{1, 2\}$ be a coloring. Then either:*

(i) there exists a monochromatic m-element subset $D \subseteq [1, 3m-2]$ with $diam(D) \geq 2m-2$, or

(ii) there exist monochromatic m-element subsets B_1, $B_2 \subseteq [1, 3m-2]$ with $B_1 <_p B_2$ and $diam(B_1) = diam(B_2) = m-1$.

Proof. Let $P = [1, m]$, and let $Q = [2m-1, 3m-2]$. From the pigeonhole principle it follows that there exists a monochromatic m-element set $Z \subseteq [1, m-1] \cup Q$. If $Z \cap [1, m-1] \neq \emptyset$ and $Z \cap Q \neq \emptyset$, then $diam(Z) \geq 2m-2$, and (i) is satisfied. Otherwise, $Q = Z$ is a monochromatic m-element set with $diam(Q) = m-1$. Repeating the argument for the set $P \cup [2m, 3m-2]$, it follows that either (i) is satisfied, or else P is a monochromatic m-element set with $diam(P) = m-1$. Hence, if in both cases (i) is not satisfied, then by letting $B_1 = Q$ and $B_2 = P$, it follows that (ii) is satisfied. $\square$

The majority of the proof consists in proving the following theorem, from which the value of $f(m, 4)$ will be shown to easily follow.

Theorem 2. *Let $m \geq 2$ be an integer, and let $\Delta : [1, 8m-6] \to \{1, 2, 3, 4\}$ be a coloring. Then either:*

(i) there exists a monochromatic m-element subset $B \subseteq [1, 8m-6]$ with $diam(B) \geq 4m-4$, or

(ii) there exist monochromatic m-element subsets B_1, $B_2 \subseteq [1, 8m-6]$ with $B_1 <_p B_2$ and $diam(B_1) \leq diam(B_2)$.

Proof. Suppose that $\Delta : [1, 8m-6] \to \{1, 2, 3, 4\}$ is a coloring such that the conclusions of the theorem do not hold. In view of (i), we can assume that:

$$\text{If } c \in \{1,2,3,4\} \text{ and } |\Delta^{-1}(c)| \geq m, \text{ then } last(c) - first(c) \leq 4m-5 \tag{1}$$

The above fact will be used frequently throughout the proof. The following statement, for which we give a short proof in the subsequent paragraph, will also be important:

$$|\Delta^{-1}(c)| < 3m-2 \text{ for every } c \in \{1,2,3,4\} \tag{2}$$

Suppose that for some $c \in \{1,2,3,4\}$, say 1, $|\Delta^{-1}(1)| \geq 3m-2$. Let B_1 be the first m integers of $\Delta^{-1}(1)$, and let B_2 consist of the next $m-1$ integers of $\Delta^{-1}(1)$ and $last(1)$. Applying (1) to color 1, it follows that $last(1) - first(1) \leq 4m-5$. Hence, since $|\Delta^{-1}(1)| \geq 3m-2$, it follows that at most $m-2$ integers in $[first(1), last(1)]$ are not colored by 1. Consequently, it follows that $diam(B_1) \leq m + (m-2) - 1 = 2m-3$. Moreover, since $|\Delta^{-1}(1)| \geq 3m-2$, it follows that $diam(B_2) \geq (3m-2) - m - 1 = 2m-3$. Thus the sets B_1 and B_2 satisfy (ii), a contradiction.

Let k be the number of colors $c \in \{1,2,3,4\}$ for which $|\Delta^{-1}(c)| \geq m$. Clearly $1 \leq k \leq 4$. We will proceed by considering three cases, each dealing with a different value that k can take.

Case 1: $k \leq 2$

Let 3 and 4 be the colors such that $|\Delta^{-1}(3)| < m$ and $|\Delta^{-1}(4)| < m$. Hence $|\Delta^{-1}(1) \cup \Delta^{-1}(2)| \geq (8m-6) - 2(m-1) = 6m-4$, yielding that $|\Delta^{-1}(1)| \geq 3m-2$ or $|\Delta^{-1}(2)| \geq 3m-2$, contradicting (2).

Case 2: $k = 3$

Without loss of generality we may assume that $|\Delta^{-1}(4)| \leq m-1$, that the greatest integer not colored by 4 is colored by 3, and that $first(1) < first(2)$. For the sake of clarity, we will divide this case into twelve steps.

STEP 1: $first(3) \geq 3m$

Since $|\Delta^{-1}(4)| \leq m-1$, and since the greatest integer not colored by 4 is colored by 3, it follows that $last(3) \geq (8m-6)-(m-1) = 7m-5$. Hence from (1) it follows that $first(3) \geq last(3) - (4m-5) \geq 3m$.

STEP 2: There exists an $\alpha \leq m-1$, such that $\alpha = |\Delta^{-1}(4) \cap [6m-5-\alpha,\, 8m-6]|$

Let α be the number of integers colored by 4 that are greater than $last_{2m}(\{1,2,3\})$. Since $|\Delta^{-1}(4)| \leq m-1$, it follows that $last_{2m}(\{1,2,3\}) \in [5m-4,\, 8m-6]$, that $\alpha = |\Delta^{-1}(4) \cap [6m-5-\alpha,\, 8m-6]|$, that $\alpha \leq m-1$, and that $last_{2m}(\{1,2,3\}) = 6m-5-\alpha$.

STEP 3: $last(1) \leq 5m-5-\alpha$

Since $|\Delta^{-1}(4)| \leq m-1$, it follows from the definition of α (Step 2) that $|\Delta^{-1}(4) \cap [1,m]| \leq m-\alpha-1$. Hence, $min\{first(1), first(2), first(3)\} \leq m-\alpha$. Since $first(3) \geq 3m$ (Step 1), it follows that $min\{first(1), first(2)\} < first(3)$. Furthermore, in view of $first(1) < first(2)$, it follows that $min\{first(1),\, first(2),\, first(3)\} = first(1)$, and hence $first(1) \leq m-\alpha$. Consequently, it follows from (1) that $last(1) \leq (4m-5) + first(1) \leq 5m-5-\alpha$.

STEP 4: $last(2) < 6m-5-\alpha$

Suppose that $last(2) \geq 6m-5-\alpha$. Hence from (1) it follows that

$$first(2) \geq last(2) - (4m-5) \geq 2m-\alpha.$$

Hence, since $first(3) \geq 3m$ (from Step 1), it follows that $\Delta[1,\, 2m-\alpha-1] = \{1,4\}$.

From the definition of α (Step 2), it follows that $|\Delta^{-1}(4) \cap [1, 2m-1]| \leq m - \alpha - 1$. Hence $|\Delta^{-1}(1) \cap [1, 2m-\alpha-1]| \geq m$, yielding a monochromatic m-element set, $B \subseteq [1, 2m-\alpha-1]$ with $diam(B) \leq 2m - \alpha - 2$. Consequently, it follows that

$$first(2) \geq 4m - 2 \text{ and } first(3) \geq 5m + \alpha - 2, \tag{3}$$

since otherwise by letting $B_1 = B$ and letting

$$B_2 = \{\text{the first } m-1 \text{ elements of color } 2\} \cup \{last(2)\},$$

or

$$B_2 = \{\text{the first } m-1 \text{ elements of color } 3\} \cup \{last(3)\},$$

it follows, since by assumption $last(2) \geq 6m-5-\alpha$, that $diam(B_2) \geq (6m-5-\alpha)-(4m-3) = 2m - \alpha - 2$ (or, since $\alpha \leq m-1$ (from Step 2), and since the greatest integer not colored by 4 is colored by 3, that $diam(B_2) \geq (7m-5) - (5m+\alpha-3) = 2m - \alpha - 2$), and (ii) is satisfied, a contradiction. However, from (3) it follows that $\Delta[1, 4m-3] = \{1, 4\}$. Hence, since $|\Delta^{-1}(4)| \leq m-1$, it follows that $|\Delta^{-1}(1) \cap [1, 4m-3]| \geq 3m-2$, contradicting (2). So $last(2) < 6m - 5 - \alpha$.

STEP 5: (a) $\alpha > 0$; (b) there exists a monochromatic m-element set $D \subseteq [first(3),\ 8m-6]$, such that $\Delta(D) = \{3\}$ and $diam(D) \geq 2m+d$, where $d = max\{0,\ (6m-5-\alpha)-first(3)\}$; and (c) $m > 2$.

Since $last(2) < 6m - 5 - \alpha$ (Step 4), and since $last(1) \leq 5m - 5 - \alpha$ (Step 3), it follows that $\Delta[6m-5-\alpha,\ 8m-6] = \{3, 4\}$. Hence, from the definition of α (Step 2), it follows that $|\Delta^{-1}(3) \cap [6m-5-\alpha,\ 8m-6]| = 2m$. Let f be the least integer colored by color 3 in $[6m-5-\alpha,\ 8m-6]$. If there are $2m$ consecutive integers colored by 3, then by letting B_1 be the first of these $2m$ consecutive integers, and letting B_2 be the last of these $2m$ consecutive integers, it follows that B_1 and B_2 satisfy (ii), a contradiction. So there are not $2m$ consecutive integers colored by 3 implying, since there are at least $2m$ integers of color 3, that $last(3) - f \geq 2m$; and also that (a) $\alpha > 0$. Thus if we let D' consist of the first $m-1$ integers colored by 3 in $[6m-5-\alpha,\ 8m-6]$ and $last(3)$, it follows that D' is an m-element monochromatic in color 3 subset with $diam(D') \geq 2m$. By exchanging the least element of D' with $first(3)$, we obtain (b) a monochromatic m-element set, D, with $diam(D) \geq 2m + d$, where $d = max\{0,\ (6m-5-\alpha) - first(3)\}$. Note since $2m \geq 4m-4$ for $m = 2$, that the proof of Theorem 2.2 is complete unless (c) $m > 2$.

STEP 6: $first(3) \geq 4m - \alpha$

If $d \geq 2m-4$, then from Step 5 it follows that $diam(D) \geq 4m-4$, and (i) is satisfied, a contradiction. So $d \leq 2m-5$. Substituting the definition of d (Step 5) into this inequality yields, if $d \neq 0$, that $first(3) \geq (6m-5-\alpha) - (2m-5) = 4m-\alpha$. If $d = 0$, then from the definition of d (Step 5) it follows that $first(3) \geq 6m-5-\alpha$. Thus, since $4m-\alpha \leq 6m-5-\alpha$ for $m > 2$, and since $m > 2$ (Step 5), it follows that $first(3) \geq 4m-\alpha$ in the case $d = 0$ as well.

STEP 7: (a) $first(1) \leq m-\alpha-\beta-d-3$, where $\beta = |\Delta^{-1}(4) \cap [4m-\alpha-2,\ 6m-6-\alpha-d]|$; and (b) there exists a monochromatic m-element set $B \subseteq [1,\ 3m-\alpha-\beta-2]$ with $diam(B) \leq 3m-\alpha-\beta-3$

Since $first(3) \geq 4m-\alpha$ (Step 6), it follows from the definitions of α (Step 2) and

β that $\Delta[1, 3m - \alpha - \beta - 2] = \{1, 2, 4\}$ with at most $m - \alpha - \beta - 1$ integers colored by 4. Hence there are at least $(3m - \alpha - \beta - 2) - (m - \alpha - \beta - 1) = 2m - 1$ elements colored by either 1 or 2 in $[1, 3m - \alpha - \beta - 2]$. By the pigeonhole principle, it follows that (a) there exists a monochromatic m-element set $B \subseteq [1, 3m - \alpha - \beta - 2]$ either of color 1 or 2. Since $first(1) < first(2)$, it follows that if $first(1) > m - \alpha - \beta - d - 3$, then $diam(B) \leq (3m - \alpha - \beta - 2) - (m - \alpha - \beta - d - 2) = 2m + d$, and by letting $B_1 = B$ and $B_2 = D$ (from Step 5), conclusion (ii) will be satisfied, a contradiction. So (b) $first(1) \leq m - \alpha - \beta - d - 3$.

STEP 8: $5m - \alpha - \beta - d - 7 \geq 4m - \alpha - 2$

If $d > m - \alpha - \beta - 4$, then the set B (from Step 7), and the set D (from Step 5), satisfy (ii), a contradiction. So $d \leq m - \alpha - \beta - 4$, and hence $5m - \alpha - \beta - d - 7 \geq 4m - 3$. Thus since $\alpha > 0$ (Step 5), it follows that $5m - \alpha - \beta - d - 7 \geq 4m - \alpha - 2$.

STEP 9: $first(2) > 2m - \alpha - \beta - d - 2$

Since $first(1) \leq m - \alpha - \beta - d - 3$ (Step 7), it follows from (1) that $last(1) \leq first(1) + (4m - 5) = 5m - \alpha - \beta - d - 8$. Hence from the definition of d (Step 5), it follows that $\Delta[5m - \alpha - \beta - d - 7, 6m - 6 - \alpha - d] = \{2, 4\}$. From the definition of β (Step 7) and from Step 8, it follows that $|\Delta^{-1}(4) \cap [5m - \alpha - \beta - d - 7, 6m - 6 - \alpha - d]| \leq \beta$, and hence $last(2) \geq 6m - 6 - \alpha - d - \beta$. Thus from (1) it follows that $first(2) > last(2) - (4m - 4) \geq 2m - \alpha - \beta - d - 2$.

STEP 10: $|\Delta^{-1}(4) \cap [2m - \alpha - \beta - d - 1, 4m - \alpha - \beta - 3]| \geq d + 1$

If there is a monochromatic m-element set, T, contained in the interval $[2m - \alpha -$

$\beta - d - 1, 4m - \alpha - \beta - 3]$, then it will have $diam(T) \leq (4m - \alpha - \beta - 3) - (2m - \alpha - \beta - d - 1) = 2m + d - 2$. Hence, since $first(3) \geq 4m - \alpha$ (from Step 6), it follows that by letting $B_1 = T$ and $B_2 = D$ (from Step 5), conclusion (ii) will be satisfied, a contradiction. So there are at most $2(m-1)$ integers colored by 1 or 2 in the interval $[2m - \alpha - \beta - d - 1, 4m - \alpha - \beta - 3]$. Consequently, since $first(3) \geq 4m - \alpha$ (Step 6), it follows that $|\Delta^{-1}(4) \cap [2m - \alpha - \beta - d - 1, 4m - \alpha - \beta - 3]| \geq d + 1$.

STEP 11: $|\Delta^{-1}(1) \cap [1, 2m - \alpha - \beta - d - 2]| \geq m$

From Step 10, and from the definitions of α (Step 5) and β (Step 7), it follows that $|\Delta^{-1}(4) \cap [1, 2m - \alpha - \beta - d - 2]| \leq (m-1) - \alpha - \beta - (d+1) = m - \alpha - \beta - d - 2$. Hence, since $first(3) \geq 4m - \alpha$ (Step 6), and since $first(2) > 2m - \alpha - \beta - d - 2$ (Step 9), it follows that $|\Delta^{-1}(1) \cap [1, 2m - \alpha - \beta - d - 2]| \geq (2m - \alpha - \beta - d - 2) - (m - \alpha - \beta - d - 2) = m$.

STEP 12: Contradiction

From Step 11, it follows that there is an m-element monochromatic in color 1 set, $B_1 \subseteq [1, 2m - \alpha - \beta - 2]$, with $diam(B_1) \leq 2m - \alpha - \beta - 3 < 2m + d$. Thus B_1 and D (from Step 5) satisfy (ii), our final contradiction.

Case 3: $k = 4$

Without loss of generality let $\Delta(1) = 1$. Hence from (1) it follows that

$$last(1) \leq 4m - 4. \tag{4}$$

Without loss of generality let $\Delta(8m-6) = 2$. Let $\beta \in [1, 8m-6]$ be the greatest integer such that $\Delta(\beta) \neq 2$.

If $\beta < 6m - 5$, then $[6m - 5, 8m - 6]$ is monochromatic in color 2. Then by letting

$B_1 = [6m-5, 7m-6]$ and letting $B_2 = [7m-5, 8m-6]$, it follows that B_1 and B_2 satisfy (ii), a contradiction. So

$$\beta \geq 6m - 5. \tag{5}$$

Without loss of generality, let $\Delta(\beta) = 3$. Hence from (1) and from (5), it follows that $first(3) \geq (6m-5) - (4m-5) = 2m$. Since $last(2) = 8m-6$, it follows from (1) that $first(2) \geq (8m-6) - (4m-5) = 4m-1$. Consequently,

$$\Delta[1,\, 2m-1] = \{1, 4\}, \tag{6}$$

and it follows from the pigeonhole principle that there is a monochromatic m-element set $B \subseteq [1, 2m-1]$ with

$$diam(B) \leq 2m - 2. \tag{7}$$

Let $\alpha \in [1, 8m-6]$ be the least integer such that $\Delta(\alpha) \neq 1$. If $\alpha > m$, then $[1,\, m]$ is a monochromatic m-element set with diameter $m-1$, but by letting $B_1 = [1, m]$, and letting B_2 be the first m elements of $\Delta^{-1}(2)$, it follows that (ii) is satisfied, a contradiction. So $\alpha \leq m$. Hence from (6) it follows that $\Delta(\alpha) = 4$, implying from (1) that $last(4) \leq (4m-5) + first(4) \leq 5m-5$. Hence, since $last(1) \leq 4m-4$ from (4), it follows that $\Delta[5m-3,\, 8m-6] = \{2, 3\}$. Translating the interval $[5m-3,\, 8m-6]$ to $[1,\, 3m-2]$ and applying Theorem 2.1, it follows that either (ii) is satisfied, a contradiction, or that there exists a monochromatic m-element subset $F \subseteq [5m-3, 8m-6]$ with $diam(F) \geq 2m-2$. In the latter case, by letting $B_1 = B$ (from (7)), and letting $B_2 = F$, conclusion (ii) follows, a contradiction. □

Theorem 3. *For an integer $m \geq 2$, $f(m, 4) = 12m - 9$*

Proof. The coloring $\Delta : [1,\, 12m - 10] \rightarrow \{1, 2, 3, 4\}$ given by the string

$$41^{m-1}2^{m-1}3^{m-1}4^{m-1}1^{m-1}2^{m-1}3^{m-1}1^{m-1}2^{m-1}3^{m-1}4^{2m-1}$$

was shown by Bialostocki, Erdős, and Lefmann [3] to imply $f(m, 4) \geq 12m - 9$.

To see $f(m, 4) \leq 12m - 9$, let $\Delta : [1, 12m - 9] \rightarrow \{1, 2, 3, 4\}$ be an arbitrary coloring. From the pigeonhole principle, it follows that $[1,\, 4m - 3]$ contains a monochromatic m-element subset, P, with $diam(P) \leq 4m - 4$. Translating the interval $[4m - 2,\, 12m - 9]$ to the interval $[1, 8m - 6]$ and applying Theorem 2.2 completes the proof. □

Acknowledgements: I would like to thank Arie Bialostocki, for introducing me to the problem and for his help with the editing of the paper.

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