On a Partition Analog of the Cauchy-Davenport Theorem*

David J. Grynkiewicz †

December 11, 2008

Abstract

Let G be a finite abelian group, and let n be a positive integer. From the Cauchy-Davenport Theorem it follows that if G is a cyclic group of prime order, then any collection of n subsets $A_1, A_2, \ldots, A_n$ of G satisfies

$$\left|\sum_{i=1}^{n} A_i\right| \geq \min\{|G|, \sum_{i=1}^{n} |A_i| - n + 1\}.$$

M. Kneser generalized the Cauchy-Davenport Theorem for any abelian group. In this paper, we prove a sequence-partition analog of the Cauchy-Davenport Theorem along the lines of Kneser's Theorem. A particular case of our theorem was proved by J. E. Olson in the context of the Erdős-Ginzburg-Ziv Theorem.

---

*Keywords: zero sum, Cauchy-Davenport; MSC: 11B75 (05D10)

†The research for this paper was done in partial fulfillment of the requirements for graduating with departmental honors in mathematics at Bates College, Lewiston, ME.

1 Introduction and Main Result

Let $(G, +, 0)$ be an abelian group. If $A, B \subseteq G$, then their *sumset*, $A + B$, is the set of all possible pairwise sums, i.e. $\{a + b \mid a \in A, b \in B\}$. A set $A \subseteq G$ is *H_a-periodic*, if it is the union of H_a-cosets for some subgroup H_a of G (note this definition of periodic differs slightly from the usual definition by allowing H_a to be trivial). A set which is maximally H_a-periodic, with H_a the trivial group, is *aperiodic*, and otherwise we refer to A as *nontrivially periodic*. For notational convenience, we use $\phi_a : G \to G/H_a$ to denote the natural homomorphism. If S is a sequence of elements from G, then an *n-set partition* of S is a collection of n nonempty subsequences of S, pairwise disjoint as sequences, such that every term of S belongs to exactly one of the subsequences, and the terms in each subsequence are all distinct. Thus such subsequences can be considered as sets. Let $A = (A_n, \ldots, A_1)$ be an n-set partition of a sequence S of elements from G whose sumset (i.e. the sumset of whose terms) is H_a-periodic. Let $y \in G/H_a$. If $y \in \phi_a(A_i)$ for all i, then y is an *H_a-nonexception*, and otherwise y is an *H_a-exception*. If $|\phi_a^{-1}(y) \cap A_j| \geq 2$, then y is an *H_a-doubled* element. The number of $y \in G/H_a$ that are H_a-nonexceptions of A is denoted by $N(A, H_a)$. The number of terms x of S such that $\phi_a(x)$ is an H_a-exception of A is denoted by $E(A, H_a)$. Finally, $|S|$ denotes the cardinality of S, if S is a set, and the length of S, if S is a sequence.

We begin with the classical theorem of Kneser for sumsets [41, 42, 36, 40, 43].

Kneser's Theorem. *Let G be an abelian group, and let $A_1, A_2, \ldots, A_n$ be a collection of finite subsets of G. If $\sum_{i=1}^{n} A_i$ is maximally H_a-periodic, then*

$$\left|\sum_{i=1}^{n} \phi_a(A_i)\right| \geq \sum_{i=1}^{n} |\phi_a(A_i)| - n + 1.$$

Observe that if $A+B$ is maximally H_a-periodic and $\rho = |A+H_a| - |A| + |B+H_a| - |B|$ is the number of 'holes' in A and B, then Kneser's Theorem implies $|A+B| \geq |A| + |B| - |H_a| + \rho$. Consequently, if either A or B contains a unique element from some H_a-coset, then $|A+B| \geq |A|+|B|-1$. In the special case that $|G|$ is prime, Kneser's Theorem reduces to the Cauchy-Davenport Theorem [25, 43].

Cauchy-Davenport Theorem. *For positive integers n and p, let $A_1, \ldots, A_n \subseteq \mathbb{Z}_p$. If p is prime, then*

$$|\sum_{i=1}^{n} A_i| \geq \min\{p, \sum_{i=1}^{n} |A_i| - n + 1\}.$$

The main result of this paper is the following, from which Corollary 1 will be derived.

Theorem 1. *Let S be a finite sequence of elements from an abelian group G, let $A = (A_n, \ldots, A_1)$ be an n-set partition of S, and let $a_i \in A_i$ for $i \in \{1, \ldots, n\}$. Then there exists an n-set partition $A' = (A'_n, \ldots, A'_1)$ of S with sumset H_a-periodic, $\sum_{i=1}^{n} A_i \subseteq \sum_{i=1}^{n} A'_i$, $a_i \in A'_i$ for $i \in \{1, \ldots, n\}$, and*

$$\left|\sum_{i=1}^{n} A'_i\right| \geq \big(E(A', H_a) + (N(A', H_a) - 1)n + 1\big)\, |H_a|.$$

Corollary 1. *Let n be a positive integer, and let S be a sequence of elements from an abelian group G of order m, such that $|S| \geq n$ and every element of S appears at most n times in S. Furthermore, let p be the smallest prime divisor of m. Then either:*

(i) there exists an n-set partition $(A_n, \ldots, A_1)$ of S such that:

$$\left|\sum_{i=1}^{n} A_i\right| \geq \min\{m,\, (n+1)p,\, |S| - n + 1\}, \text{ or}$$

(ii) (a) there exists $\alpha \in G$ and nontrivial proper $H_a \leq G$, with $[G : H_a] = a$, such that all but at most $a - 2$ terms are from the coset $\alpha + H_a$, and (b) there exists an n-set partition, $(A_n, \ldots, A_1)$, of the subsequence of S consisting of all the terms that belong to the coset $\alpha + H_a$, such that

$$\sum_{i=1}^{n} A_i = n\alpha + H_a.$$

The meaning of Theorem 1 and Corollary 1 may not be clearly evident upon first reading, so it is worthwhile to describe them in less rigorous language, including historical context. In 1961, Erdős, Ginzburg, and Ziv proved that any sequence of $2m-1$ elements from an abelian group of order m contains an m-term subsequence the sum of whose terms is zero (i.e. contains an m-term zero-sum subsequence) [26]. If a sequence with elements from the cyclic group $\mathbb{Z}_m$ consists entirely of $0's$ and $1's$, then its m-term monochromatic subsequences correspond precisely with its m-term zero-sum subsequences. Hence the aforementioned Erdős-Ginzburg-Ziv Theorem can thought of as a generalization of the pigeonhole principle for m pigeons and 2 holes. The result led to the development of zero-sum Ramsey theory, in much the same way as the pigeonhole principle lies at the base of the development of ordinary Ramsey theory. Any Ramsey theory problem has a corresponding zero-sum version obtained by replacing colorings using two colors with colorings using the elements of $\mathbb{Z}_m$ and looking for zero-sum substructures rather than monochromatic ones. Through a slightly more complicated way, Ramsey theory questions involving more than two colors also have a corresponding zero-sum version [7, 34, 35]. If m is chosen to be the size of the particular substructure in question, then the zero-sum Ramsey number always gives an upper bound on the monochromatic Ramsey number, but, perhaps unexpectedly, the two numbers were in many cases equal. Such problems are said to zero-sum generalize. Examples of particular problems include [4, 5, 6, 7, 8, 9, 11, 15, 20, 21, 45] the most well known of which is the zero-trees theorem [29, 46]. However, problems of this type proved to be difficult. Modifying the condition so that m need only divide the size of the particular substructure in question, leads to a different flavor of problem, for which more progress has been accomplished [1, 12, 13, 14, 16, 17, 18, 19, 22, 23, 24, 38, 39, 45].

However, as plagues many areas of combinatorics and in particular additive number theory, there are very few tools for tackling problems of these types that do not rely on m being prime. Often the only hope for obtaining a composite result lies in finding a clever argument to derive the composite cases from the prime via induction on the number of prime factors. However, depending on the particular problem, this may be quite difficult to accomplish. The Cauchy-Davenport Theorem is one tool that is sometimes effective for dealing with a zero-sum question, but which is only valid for m prime. An equivalent rephrasing of the Cauchy-Davenport Theorem is that the sumset of *every* n-set partition of a subsequence S of elements from $\mathbb{Z}_p$ contains at least $\min\{p, |S| - n + 1\}$ elements. However, in many applications it is sufficient to know simply that there *exists* some n-set partition of S with this property.

Note that if either $|S| \leq (n+1)p + n - 1$ or $n \geq \frac{m}{p} - 1$ then the inequalities in Corollary 1 reduce to the bound given in Cauchy-Davenport. Thus in loose terms, under the previously stated conditions Corollary 1 says that the existence version of the Cauchy-Davenport Theorem is valid for any G, not just $\mathbb{Z}_p$, except when S is essentially (i.e. with very few exceptions) a sequence of terms from some smaller nontrivial subgroup translate $\alpha + H_a$ of G with the existence result then holding modulo H_a. But under these restrictive conditions it follows, from the Erdős-Ginzburg-Ziv Theorem, that any subsequence of S with length $m + \frac{m}{a} + a - 3$ must contain an m-term zero-sum subsequence. Since $m + \frac{m}{a} + a - 3 \leq \lfloor \frac{3}{2} m \rfloor - 1$, this is a often a sufficiently significant improvement over the Erdős-Ginzburg-Ziv Theorem. We remark that the case $|S| = 2m - 1$ and $n = m$ in Corollary 1 reduces to a Theorem originally proven by Olson [44], whose result gave one of the first existence-type lower bounds for an n-set partition. A related n-set partition theorem for products of two primes was used by Furedi and Kleitman to estimate the number of zero-sum subsequences [28].

However, in understanding how Corollary 1 can be applied to zero-sum questions, it is best to refer to several examples. The following references also contain several theorems that are often quite useful when used in conjunction with Corollary 1. Corollary 1 was used to establish a four-color zero-sum generalization that was the next open case in a conjecture of Bialostocki, Erdős, and Lefmann [7, 35]. Relaxing the structure from the aforementioned problem, a zero-sum generalization was obtained, with the aid of the Olson case of Corollary 1, for two, three, four and five colors [34]. In [33], the structure considered by Bialostocki, Erdős, and Lefmann was generalized to a larger family, and Corollary 1 was used to give linear bounds on the corresponding two color zero-sum Ramsey numbers, and also, in one instance, to determine the zero-sum number exactly. Let $g(m, k)$ denote the least integer n such that any sequence of elements from $\mathbb{Z}_m$ with length n and k distinct residues must contain an m-term zero-sum subsequence. The function $g(m, k)$ was introduced by Bialostocki and Lotspeich [10], and has been studied by several authors, [9, 27, 30, 31, 37]. The value of $g(m, k)$ was known completely for $k \leq 4$, [10]. Using Corollary 1, $g(m, k)$ was completely determined for the case $k = 5$, [2]. Finally, Corollary 1 was also used to help extend the Erdős-Ginzburg-Ziv Theorem from a single edge to a class of hypergraphs [3, 32].

The assertion of Theorem 1, which is the more general statement of the application phrased Corollary 1, is quite natural. From Kneser's Theorem we know that if a given n-set partition $A = (A_n, \ldots, A_1)$ fails to satisfy the Cauchy-Davenport bound, then its sumset must be nontrivially H_a-periodic. If H_a is maximal, then modulo H_a the sumset of A is aperiodic. Thus if in some set A_i of A there are two elements from the same H_a-coset, and if there is some set A_j of A that does not contain an element from this coset, then we know that the bound given by Kneser's Theorem on the cardinality of the sumset of A modulo H_a will increase by moving one of the two elements from A_i to A_j. It is natural to think the sumset (not modulo H_a) will likewise increase, and thus repeating this moving procedure we should be able to attain the Cauchy-Davenport bound unless a small number of H_a-cosets contain most of the terms of S. Theorem 1 asserts that this is essentially true. The Cauchy-Davenport bound asserts that each term of S partitioned by the n-set partition A—minus one term per A_j with $j \geq 2$ that instead transfers all elements in the sumset $\sum_{i=1}^{j-1} A_i$ to the sumset $\sum_{i=1}^{j} A_i$—contributes at least one element to the sumset $\sum_{i=1}^{n} A_i$. Theorem 1 says that there exists an n-set partition A of S with sumset H_a-periodic such that, if we equate all terms of S that both belong to the same H_a-nonexception and are also contained in the same set A_i, then each resulting term of S, minus one resulting term per A_j with $j \geq 2$, contributes at least one H_a-coset to the sumset $\sum_{i=1}^{n} A_i$. Hence only terms of S that belong to an H_a-nonexception will contribute to any deficit between the Cauchy-Davenport bound and the actual cardinality of $\sum_{i=1}^{n} A_i$.

We conclude the introduction with the following simple proposition needed for the proof of Theorem 1 [Theorem 3.2, 40].

Proposition 1. *If A and B are finite subsets of an abelian group, $b \in B$, and $A + B \neq A + (B \setminus \{b\})$, then $|A + B| \geq |A| + |B| - 1$.*

2 Proof of Theorem 1

The proof of Theorem 1 is constructive in nature, and will be presented as a series of lemmas. In what follows, S is a finite sequence of elements from an abelian group G, and n is a fixed positive integer, and $A = (A_n, \ldots, A_1)$ is an n-set partition of S that by contradiction does not satisfy Theorem 1. The proof makes use of an n-set partition that satisfies a list of iterated extremal conditions that are rigorously described by the following two lengthy definitions.

Definition 1. For a fixed integer $r \leq n$, an *r-maximal partition set* of S, denoted by Λ_r, is the set consisting of those ordered n-set partitions of S which can be constructed recursively by the method described below. For the sake of clarity, in addition to Λ_i, we introduce four associated entities denoted by $\mathcal{F}_i$, Υ_i, $\mathcal{G}_i$, and $H_{k_{i+1}}$, for $i = 0, \ldots, r-1$.

Λ_0 consists of all ordered n-set partitions, $(Z_n, \ldots, Z_1)$, of S such that $\sum_{i=1}^{n} A_i \subseteq \sum_{i=1}^{n} Z_i$ and $a_i \in Z_i$ for $i \leq n$.

$\mathcal{F}_0 = (A_n^0, \ldots, A_1^0)$ is a fixed element of Λ_0.

Υ_0 is the subset of Λ_0 consisting of all ordered n-set partitions, $(Z_n, \ldots, Z_1)$, for which $|\sum_{i=1}^{n} Z_i|$ is maximum.

$\mathcal{G}_0 = (B_n^0, \ldots, B_1^0)$ is a fixed element of Υ_0. Different choices of $\mathcal{G}_0$ may result in different Λ_r's.

H_{k_1} is the maximal subgroup for which the sumset $\sum_{i=1}^{n} B_i^0$ is periodic.

Suppose Λ_{j-1}, $\mathcal{F}_{j-1} = (A_n^{j-1}, \ldots, A_1^{j-1})$, Υ_{j-1}, $\mathcal{G}_{j-1} = (B_n^{j-1}, \ldots, B_1^{j-1})$, and H_{k_j} have been constructed; then we proceed as follows:

$$\Lambda_j = \left\{ (Z_n, \ldots, Z_1) \in \Upsilon_{j-1} \mid \sum_{i=1}^{n} |\phi_{k_j}(Z_i)| \text{ is maximum subject to } \sum_{i=j}^{n} Z_i = \sum_{i=j}^{n} B_i^{j-1} \right\}.$$

$\mathcal{F}_j = (A_n^j, \ldots, A_1^j)$ is a fixed element of Λ_j. Different choices of $\mathcal{F}_j$ may result in different Λ_r's.

$$\Upsilon_j = \left\{ (Z_n, \ldots, Z_1) \in \Lambda_j \mid \left| \sum_{i=j+1}^{n} Z_i \right| \text{ is maximum subject to } \phi_{k_j}(A_i^j) \subseteq \phi_{k_j}(Z_i) \text{ for all } i \right\}.$$

$\mathcal{G}_j = (B_n^j, \ldots, B_1^j)$ is a fixed element of Υ_j. Different choices of $\mathcal{G}_j$ may result in different Λ_r's.

$H_{k_{j+1}}$ is the maximal subgroup for which the sumset $\sum_{i=j+1}^{n} B_i^j$ is periodic.

Definition 2. For a fixed integer ρ, where $0 \le \rho \le n-2$, a *ρ-factor form* of S is an ordered n-set partition of S, say $F_\rho = (Z_n, \ldots, Z_1) = (Y_n, \ldots, Y_{\rho+1}, X_\rho, \ldots, X_1)$, which satisfies:

(I) if $1 \le j \le \rho+1$, then $\sum_{i=j}^{n} Z_i$ is maximally H_{k_j}-periodic with H_{k_j} a proper nontrivial subgroup—for simplicity we will sometimes denote $k_{\rho+1}$ by k;

(II) $|\sum_{i=1}^{n} \phi_k(Z_i)| \ge |\sum_{i=\rho+1}^{n} \phi_k(Y_i)| + \sum_{i=1}^{\rho} |\phi_k(X_i)| - (\rho+1) + 1$;

(III) $|\sum_{i=\rho+1}^{n} Y_i| < \sum_{i=\rho+1}^{n} |Y_i| - (n-\rho) + 1$;

(IV) each term X_i, for all $i \le \rho$, contains an element mapped to an H_{k_i}-exception;

(V) there exists a $(\rho+1)$-maximal partition set $\Lambda_{\rho+1}$ of S, such that $F_\rho \in \Lambda_{\rho+1}$.

If F_ρ is an ordered n-set partition of S that satisfies (I), (IV) and (V), then it is called a *weak ρ-factor form*. It should be noted that (I) easily implies that $H_{k_{j+1}} \le H_{k_j}$ for $j \in \{1, \ldots, \rho\}$.

Lemma 1. *If S has a weak ρ-factor form, $F_\rho = (Z_n, \ldots, Z_1) = (Y_n, \ldots, Y_{\rho+1}, X_\rho, \ldots, X_1)$, such that for some index q there exists $x \in Z_q$, where $\phi_k(x)$ is an H_k-doubled H_k-exception, then $q \ge \rho+1$ and $\sum_{\substack{i=\rho+1 \\ i \ne q}}^{n} Z_i + (Z_q \setminus \{x\})$ is not H_k-periodic.*

Proof. Since $\phi_k(x)$ is doubled, it follows that there are at least two elements of Z_q mapped by ϕ_k to $\phi_k(x)$. Hence w.l.o.g. we may assume $x \ne a_q$. Let $l = \min\{\rho+1, q\}$. From the definition of an exception, it follows that there must exist a term D of F_ρ such that $\phi_k(x) \notin \phi_k(D)$. Suppose $\sum_{\substack{i=l \\ i \ne q}}^{n} Z_i + (Z_q \setminus \{x\})$ is still H_k-periodic. Then by (I) and the definition of a doubled element, it follows that $\sum_{i=l}^{n} Z_i = \sum_{\substack{i=l\\ i\neq q}}^{n} Z_i + (Z_q \setminus \{x\})$. Hence, since $x \neq a_q$, it follows that if we remove x from Z_q and place it in D, we obtain a new ordered n-set partition, say $F'_\rho = (Z'_n, \ldots, Z'_1)$, of S such that

$$\sum_{i=j}^{n} Z_i \subseteq \sum_{i=j}^{n} Z'_i \text{ for every } j \leq \rho + 1. \tag{1}$$

Since $\phi_k(x)$ is doubled and since $\phi_k(x) \notin \phi_k(D)$, it follows that

$$\sum_{i=1}^{n} |\phi_k(Z'_i)| > \sum_{i=1}^{n} |\phi_k(Z_i)|. \tag{2}$$

By (V) and the definition of an r-maximal partition set, it follows that $F_\rho \in \Lambda_{\rho+1} \subseteq \Upsilon_l$, for every $l \leq \rho$. Hence in view of (1), since $\phi_k(x)$ is a doubled exception in F_ρ, since $H_k \leq H_{k_i}$ for all i, and since $F_\rho \in \Lambda_{\rho+1}$, it follows by a simple inductive argument passing from j to $j+1$ where $j = 0, \ldots, \rho$, that $F'_\rho \in \Upsilon_j$ and $F'_\rho \in \Lambda_{j+1}$. Consequently, $F'_\rho \in \Lambda_{\rho+1}$. Since $F_\rho \in \Lambda_{\rho+1}$ and since $F'_\rho \in \Lambda_{\rho+1}$, from the definition of an r-maximal partition set it follows that $\sum_{i=1}^{n} |\phi_k(Z'_i)| = \sum_{i=1}^{n} |\phi_k(Z_i)|$, contradicting (2). So $\sum_{\substack{i=l\\ i\neq q}}^{n} Z_i + (Z_q \setminus \{x\})$ is not H_k-periodic. Hence it follows from (I) that $q \geq \rho + 1$, whence $l = \rho + 1$, completing the proof of Lemma 1.□

Lemma 2. *If S has a weak ρ-factor form $F_\rho = (Z_n, \ldots, Z_1) = (Y_n, \ldots, Y_{\rho+1}, X_\rho, \ldots, X_1)$, which satisfies (III), and for which for some index q there exists $x \in Z_q$, where $\phi_k(x)$ is an H_k-doubled H_k-exception, then $|\sum_{i\neq q} Y_i| < \sum_{i\neq q} |Y_i| - (n - \rho - 1) + 1$.*

Proof. From Lemma 1, Proposition 1 and (I), it follows that $q \geq \rho + 1$ and

$$|\sum_{i\neq q} Y_i + Y_q| \geq |\sum_{i\neq q} Y_i| + |Y_q| - 1. \tag{3}$$

If the conclusion of the lemma is false, then (3) implies $|\sum_{i=\rho+1}^{n} Y_i| \geq \sum_{i=\rho+1}^{n} |Y_i| - (n - \rho) + 1$, contradicting (III).□

Lemma 3. *If S has a weak ρ-factor form $F_\rho = (Z_n, \ldots, Z_1) = (Y_n, \ldots, Y_{\rho+1}, X_\rho, \ldots, X_1)$, which satisfies (II), then F_ρ is a ρ-factor form.*

Proof. Note that we need only show that (III) holds. From Lemma 1 it follows that there cannot exist a term X_r of F_ρ, where $r \leq \rho$, such that $\phi_k(X_r)$ contains an H_k-doubled H_k-exception. Hence, since $H_k \leq H_{k_j}$ for all j, then from (IV) it follows that each term X_r with $r \leq \rho$ contains a unique element from some H_k-coset. Thus, if (III) does not hold, then it follows from (I) and (II) that Theorem 1 holds with the trivial group, contrary to assumption.□

Lemma 4. *If S has a weak ρ-factor form $F_\rho = (Z_n, \ldots, Z_1) = (Y_n, \ldots, Y_{\rho+1}, X_\rho, \ldots, X_1)$, then F_ρ is a ρ-factor form.*

Proof. From Lemma 3 we see it suffices to show that (II) holds. To this end, note that it suffices to show

$$|\sum_{i=j+1}^{n} \phi_k(Z_i) + \phi_k(X_j)| \geq |\sum_{i=j+1}^{n} \phi_k(Z_i)| + |\phi_k(X_j)| - 1, \text{ for all } j \leq \rho. \tag{4}$$

Let $j \leq \rho$ be arbitrary. From (IV) it follows that there exists $x \in X_j$ such that $\phi_{k_j}(x)$ is an H_{k_j}-exception. Suppose $\phi_{k_j}(x)$ is H_{k_j}-doubled. Then w.l.o.g. $x \neq a_j$. If $\sum_{i=j+1}^{n} \phi_k(Z_i) + \phi_k(X_j) \neq \sum_{i=j+1}^{n} \phi_k(Z_i) + (\phi_k(X_j) \setminus \{\phi_k(x)\})$, then (4) follows from Proposition 1. Otherwise, it follows that we can remove x from X_j and place it in some term D with $\phi_{k_j}(x) \notin D$, yielding a contradiction by the arguments used in the proof of Lemma 1. So we may assume $\phi_{k_j}(x)$ is not H_{k_j}-doubled. Hence it follows that $\phi_k(x)$ is the only element from its H_{k_j}/H_k-coset in $\phi_k(X_j)$. From (I) it follows that $\sum_{i=j+1}^{n} \phi_k(Z_i) + \phi_k(X_j)$ is maximally H_{k_j}/H_k-periodic. Hence (4) follows from Kneser's Theorem and the conclusions of the last two sentences.□

Lemma 5. *If S has a ρ-factor form $F_\rho = (Z_n, \ldots, Z_1) = (Y_n, \ldots, Y_{\rho+1}, X_\rho, \ldots, X_1)$, then S has a $(\rho + 1)$-factor form.*

Proof. From Lemma 4 it suffices to show S has a weak $(\rho+1)$-factor form. Since Theorem 1 does not hold with H_k, it follows from (II), (I) and Kneser's Theorem that there is an H_k-doubled H_k-exception. However, by Lemma 1 it follows that no term Z_i with $i \leq \rho$ can contain an element mapped to an H_k-doubled H_k-exception. Hence, there exists a term Y_q, such that $\phi_k(Y_q)$ contains an H_k-doubled H_k-exception. Since the order of terms Y_i for $i > \rho$ is inconsequential, we may assume w.l.o.g. that $q = \rho+1$. Define $\Upsilon_{\rho+1}$ to be

$$\Upsilon_{\rho+1} \overset{def}{=} \left\{ (Z'_n, \ldots, Z'_1) \in \Lambda_{\rho+1} \mid \left| \sum_{i=\rho+2}^{n} Z'_i \right| \text{ is maximum subject to } \phi_k(Z_i) \subseteq \phi_k(Z'_i) \text{ for all } i \right\},$$

and let

$$F'_\rho = (Z'_n, \ldots, Z'_1) = (Y'_n, \ldots, Y'_{\rho+1}, X'_\rho, \ldots, X'_1),$$

be an arbitrarily chosen element of $\Upsilon_{\rho+1}$. Note that since (V) implies $F_\rho \in \Lambda_{\rho+1}$, and since $F'_\rho \in \Lambda_{\rho+1}$, it follows that (I), (IV) and (V) hold for F'_ρ. Hence by Lemma 4 it follows that F'_ρ is a ρ-factor form.

Next we will show the inequality

$$|\sum_{i=\rho+2}^{n} Y'_i| < \sum_{i=\rho+2}^{n} |Y'_i| - (n-\rho-1) + 1. \tag{5}$$

From the definitions of $\Lambda_{\rho+1}$ and $\Upsilon_{\rho+1}$ it follows that $\phi_k(Z_i) = \phi_k(Z'_i)$ for all i. Hence, since $\phi_k(Z_{\rho+1})$ contained an H_k-exception, it follows that $\phi_k(Z'_{\rho+1})$ still contains an H_k-exception, say $\phi_k(x)$ where $x \in Z'_{\rho+1}$. If $\phi_k(x)$ is H_k-doubled, then Lemma 2 implies (5). Hence we may assume that x is the unique element from its H_k-coset in $Z'_{\rho+1}$. Thus from (I) and Kneser's Theorem, it follows that $|\sum_{i=\rho+1}^{n} Y'_i| \geq |\sum_{i=\rho+2}^{n} Y'_i| + |Y'_{\rho+1}| - 1$. Hence, if (5) does not hold, then from (I), (II) and (IV), it follows, as it did in the proof of Lemma 3, that Theorem 1 holds with the trivial group, contrary to assumption. So (5) does hold as desired. Consequently, $\rho + 1 \leq n - 2$. Furthermore, by Kneser's Theorem it follows from (5) that $\sum_{i=\rho+2}^{n} Y'_i$ is maximally $H_{k_{\rho+2}}$-periodic, with $H_{k_{\rho+2}}$ a proper nontrivial subgroup.

We can further assume w.l.o.g. that we chose F'_ρ so that $\sum_{i=1}^{n}|\phi_{k_{\rho+2}}(Z'_i)|$ is maximum with respect to all $(Z''_n,\ldots,Z''_1) \in \Upsilon_{\rho+1}$ with $\sum_{i=\rho+2}^{n} Z''_i = \sum_{i=\rho+2}^{n} Z'_i$. Thus the n-set partition $(Y'_n, Y'_{n-1},\ldots,Y'_{\rho+2}, Z'_{\rho+1}, X'_\rho,\ldots,X'_1)$ satisfies all conditions for a weak $(\rho+1)$-factor form with $Z'_{\rho+1} = X'_{\rho+1}$.□

Proof Theorem 1. Let $A' = (A'_n,\ldots,A'_1)$ be an n-set partition of S whose sumset has maximal cardinality subject to $\sum_{i=1}^{n} A_i \subseteq \sum_{i=1}^{n} A'_i$ and $a_i \in A'_i$ for $i \le n$. Since we have assumed Theorem 1 does not hold for A, it follows that (III) holds with $\rho = 0$. Hence from Kneser's Theorem it follows that $\sum_{i=1}^{n} A'_i$ is maximally H_{k_1}-periodic with H_{k_1} a proper nontrivial subgroup. Thus the set partition A' satisfies (I), (II), (III) and (IV) for $\rho = 0$, and we may assume that A' has been chosen such that $\sum_{i=1}^{n}|\phi_{k_1}(A'_i)|$ is maximum over all n-set partitions $(Z_n,\ldots,Z_1)$ of S with $\sum_{i=1}^{n} Z_i = \sum_{i=1}^{n} A'_i$ and $a_i \in A'_i$ for all i. Thus the sequence S has a 0-factor form given by the n-set partition A'. Let γ be the maximum integer for which S has a γ-factor form; it follows that γ exists, since ρ is bounded from above by $n-2$ from the definition of a ρ-factor form. Now it follows from Lemma 5 that S has a $(\gamma+1)$-factor form, contradicting the maximality of γ.□

Proof Corollary 1. We use induction on $|S|$. Since $|S| \ge n$ by hypothesis, the base case is $|S| = n$; and it is trivially seen to satisfy (i). Assume that Corollary 1 holds for all sequences S' with $|S'| < |S|$. Note that the conditions of the hypothesis imply that S has an n-set partition A. Applying Theorem 1 to A yields (i) unless $N(A,H_a) = 1$, $r_1 \stackrel{def}{=} E(A,H_a) \le a-2$ and $(ii)(a)$ holds with H_a, where $[G : H_a] = a$. Furthermore, Theorem 1 and $N(A,H_a) = 1$ imply $(ii)(b)$ unless $r_1 > 0$. Hence,

$$2\frac{m}{a} + n - 1 \le |S'| < |S|, \tag{6}$$

where S' is the subsequence of S consisting of all terms mapped to the H_a-nonexception, as otherwise Theorem 1 implies (i). Let α be an arbitrary element of S', and let S'' be the sequence obtained by adding $-\alpha$ to every term of S'. Since the terms of S'' belong to

H_a, and since $|S''| < |S|$ by (6), it follows from the induction hypothesis that we can apply Corollary 1 to S'', yielding either (i) or (ii).

If (i) holds for S'', then it is easy to induce an n-set partition $A_n, \ldots, A_1$ of S' with $|\sum_{i=1}^{n} A_i| \geq \min\{\frac{m}{a},\, (n+1)p',\, |S'| - n + 1\}$, where p' is the smallest prime divisor of $\frac{m}{a}$. We consider two subcases. (a) If $|\sum_{i=1}^{n} A_i| \geq \min\{\frac{m}{a},\, |S'| - n + 1\}$, then (6) implies that $(ii)(b)$ holds with the subgroup H_a, and the proof is complete. (b) If $|\sum_{i=1}^{n} A_i| \geq (n+1)p'$, then we will induce an n-set partition of S whose sumset contains $\sum_{i=1}^{n} A_i$. Let T be the subsequence of S obtained by deleting all terms of S' from S. Since T is a subsequence of S, and since S has a n-set partition, it follows that T has an n'-set partition $B_1, \ldots, B_{n'}$ for some $n' \leq n$. By the definition of S' it follows that no term of T appears in S', and hence the collection of sets C_i, for $i = 1, \ldots, n$, given by $C_i = A_i \cup B_i$, where $B_i = \emptyset$ for $i > n'$, is an n-set partition of S. Since $A_i \subseteq C_i$ for all i, and since $p' \geq p$, and since $|\sum_{i=1}^{n} A_i| \geq (n+1)p'$, it follows that $|\sum_{i=1}^{n} C_i| \geq (n+1)p$, yielding (i).

So we may assume (ii) holds for S'' with corresponding subgroup, say H_{ka}, where $[H_a : H_{ka}] = k$. Hence there are r_2 terms of S' mapped to H_{ka}-exceptions, where r_2 satisfies

$$r_2 \leq k - 2 \tag{7}$$

Since $r_1,\, r_2 \geq 0$, it follows that

$$(r_2 + 2)(r_1 + 1) > (r_1 + r_2 + 1).$$

Hence by (7) it follows that

$$k(r_1 + 1)\frac{m}{ka} > (r_1 + r_2 + 1)\frac{m}{ka}. \tag{8}$$

Note that from the conclusion of Theorem 1 it follows that

$$(r_1 + 1)\frac{m}{a} < \min\{m,\, (n+1)p,\, |S| - n + 1\},$$

or else (i) holds, and consequently it follows from (8) that

$$(r_1 + r_2 + 1)\frac{m}{ka} < \min\{m,\, (n+1)p,\, |S| - n + 1\}.$$

Thus $(ii)(a)$ holds for S with H_{ka}. Since $(ii)(b)$ holds for S' with H_{ka}, it follows that $(ii)(b)$ holds for S with H_{ka}. Thus (ii) holds for S, and the proof is complete.□

Acknowledgements. I would like to thank Arie Bialostocki, for all the effort he put into helping me with the editing of the paper, and my Thesis advisor Peter Wong, for his encouragement.

References

[1] N. Alon and Y. Caro, On three zero-sum Ramsey-type problems, *J. Graph Theory*, **17** (1993), no. 2, 177–192.

[2] A. Bialostocki, P. Dierker, D. Grynkiewicz and M. Lotspiech, On some developments of the Erdős-Ginzburg-Ziv Theorem II, *Acta. Arith.*, **100** (2003), no. 2, 173–184.

[3] A. Bialostocki and D. Grynkiewicz, On the intersection of two m-sets and the Erdős-Ginzburg-Ziv Theorem, submitted (2004).

[4] A. Bialostocki, G. Bialostocki and D. Schaal, A zero-sum theorem, *J. Combin. Theory Ser. A*, **101** (2003), no. 1, 147–152.

[5] A. Bialostocki, R. Sabar and D. Schaal, On a zero-sum generalization of a variation of Schur's equation, submitted (2002).

[6] A. Bialostocki and D. Schaal, On a variation of Schur numbers, *Graphs Combin.*, **16** (2000), no. 2, 139–147.

[7] A. Bialostocki, P. Erdős and H. Lefmann, Monochromatic and zero-sum sets of nondecreasing diameter, *Discrete Math.*, **137** (1995), no. 1–3, 19–34.

[8] A. Bialostocki, Zero-sum trees: a survey of results and open problems, in *Finite and Infinite Combinatorics in Sets and Logic* (Netherlands, 1993), eds. N. W. Sauer et al, Kluwer Academic Publishers, 19–29.

[9] A. Bialostocki and P. Dierker, On the Erdős-Ginzburg-Ziv theorem and the Ramsey numbers for stars and matchings, *Discrete Math.*, **110** (1992), no. 1–3, 1–8.

[10] A. Bialostocki and M. Lotspeich, Developments of the Erdős-Ginzburg-Ziv Theorem I, in *Sets, graphs and numbers* (Budapest, 1991), 97–117.

[11] A. Bialostocki, Y. Caro and Y. Roditty, On zero sum Turan numbers, Twelfth British Combinatorial Conference (Norwich, 1989), *Ars Combin.*, **29** (1990), A, 117–127.

[12] Y. Caro and C. Provstgaard, Zero-sum delta-systems and multiple copies of graphs, *J. Graph Theory*, **32** (1999), no. 2, 207–216.

[13] Y. Caro and Y. Yuster, The characterization of zero-sum (mod 2) bipartite Ramsey numbers, *J. Graph Theory*, **29** (1998), no. 3, 151–166.

[14] Y. Caro, Binomial coefficients and zero-sum Ramsey numbers, *J. Combin. Theory Ser. A*, **80** (1997), no. 2, 367–373.

[15] Y. Caro, Zero-sum problems—a survey, *Discrete Math.*, **152** (1996), no. 1–3, 93–113.

[16] Y. Caro and Y. Roditty, A zero-sum conjecture for trees, *Ars Combin.*, **40** (1995), 89–96.

[17] Y. Caro, A complete characterization of the zero-sum (mod 2) Ramsey numbers, *J. Combin. Theory Ser. A*, **68** (1994), no. 1, 205–211.

[18] Y. Caro, A linear upper bound in zero-sum Ramsey theory, *Internat. J. Math. Math. Sci.*, **17** (1994), no. 3, 609–612.

[19] Y. Caro, Zero-sum bipartite Ramsey numbers, *Czechoslovak Math. J.*, **43(118)** (1993), no. 1, 107–114.

[20] Y. Caro and Y. Roditty, On zero-sum Turan problems of Bialostocki and Dierker, *J. Austral. Math. Soc. Ser. A*, **53** (1992), no. 3, 402–407.

[21] Y. Caro, On zero-sum Ramsey numbers—stars, *Discrete Math.*, **104** (1992), no. 1, 1–6

[22] Y. Caro, On several variations of the Turan and Ramsey numbers, *J. Graph Theory*, **16** (1992), no. 3, 257–266.

[23] Y. Caro, On zero sum Ramsey numbers—complete graphs, *Quart. J. Math. Oxford* Ser. (2), **43** (1992), no. 170, 175–181.

[24] Y. Caro, On zero-sum delta-systems and multiple copies of hypergraphs, *J. Graph Theory*, **15** (1991), no. 5, 511-521.

[25] H. Davenport, On the addition of residue classes, *J. London Math. Society*, **10** (1935), 30–32.

[26] P. Erdős, A. Ginzburg and A. Ziv, Theorem in additive number theory, *Bull. Res. Council Israel*, **10F** (1961), 41–43.

[27] C. Flores and O. Ordaz, *On the Erdős-Ginzburg-Ziv theorem*, Discrete Math., **152** (1996), no. 1-3, 321–324.

[28] Z. Furedi and D. J. Kleitman, The minimal number of zero sums, in *Combinatorics, Paul Erdos is eighty*, Vol. 1 (Budapest, 1993), Bolyai Soc. Math. Stud., János Bolyai Math. Soc., 159–172.

[29] Z. Furedi and D. Kleitman, *On zero-trees*, J. Graph Theory, **16** (1992), 107–120.

[30] L. Gallardo, G. Grekos and J. Pihko, On a variant of the Erdős-Ginzburg-Ziv problem, *Acta Arith.*, **89** (1999), no. 4, 331–336.

[31] W. D. Gao and Y. O. Hamidoune, Zero sums in abelian groups, *Combin. Probab. Comput.*, **7** (1998), no. 3, 261–263.

[32] D. Grynkiewicz, An extension of the Erdős-Ginzburg-Ziv Theorem to hypergraphs, submitted (2004).

[33] D. Grynkiewicz and R. Sabar, Monochromatic and zero-sum sets of nondecreasing modified-diameter, submitted (2003).

[34] D. Grynkiewicz and A. Schultz, A 5-color Zero-Sum Generalization, submitted (2002).

[35] D. Grynkiewicz, On four colored sets with nondecreasing diameter and the Erdős-Ginzburg-Ziv Theorem, *J. Combin. Theory Ser. A*, **100** (2002), 44–60.

[36] H. Halberstein and K. F. Roth, *Sequences*, (New York, 1983), Springer-Verlag.

[37] Y. O. Hamidoune, O. Ordaz and A. Ortuño, On a combinatorial theorem of Erdős, Ginzburg and Ziv, *Combin. Probab. Comput.*, **7** (1998), no. 4, 403–412.

[38] H. Harborth and L. Piepmeyer, Zero-sum Ramsey numbers modulo 3, *J. Combin. Theory Ser. A*, **75** (1996), no. 1, 145–147

[39] H. Harborth and L. Piepmeyer, The zero-sum Ramsey numbers $r(K_4, Z_3)$ and $r(K_6, Z_3)$, Proceedings of the Twenty-fifth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1994), *Congr. Numer.*, **101** (1994), 51–54.

[40] J. H. B. Kemperman, On small sumsets in an abelian group, *Acta Math.*, **103** (1960), 63–88.

[41] M. Kneser, Ein satz über abelsche gruppen mit anwendungen auf die geometrie der zahlen, *Math. Z.*, **64** (1955), 429–434.

[42] M. Kneser, Abschätzung der asymptotischen dichte von summenmengen, *Math. Z.*, **58** (1953), 459–484.

[43] M. B. Nathanson, *Additive Number Theory. Inverse Problems and the Geometry of Sumsets*, Graduate Texts in Mathematics, vol. 165 (New York, 1996), Springer-Verlag.

[44] J. E. Olson, An addition theorem for finite abelian groups, *Journal of Number Theory*, **9** (1977), 63–70.

[45] Y. Roditty, On zero sum Ramsey numbers of multiple copies of a graph, *Ars Combin.*, **35** (1993), A, 89–95.

[46] A. Schrijver and P. D. Seymour, A simpler proof and a generalization of the zero-trees theorem, *J. Combin. Theory Ser. A*, **58** (1991), no. 2, 301–305.

David J. Grynkiewicz

Mathematics 253-37

Caltech

Pasadena, CA 91125

USA

diambri@hotmail.com