# A Weighted Erdős-Ginzburg-Ziv Theorem

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**Abstract**

An *n-set partition* of a sequence $S$ is a collection of $n$ nonempty subsequences of $S$, pairwise disjoint as sequences, such that every term of $S$ belongs to exactly one of the subsequences, and the terms in each subsequence are all distinct so that they can be considered as sets. If $S$ is a sequence of $m+n-1$ elements from a finite abelian group $G$ of order $m$ and exponent $k$, and if $W=\{w_i\}_{i=1}^n$ is a sequence of integers whose sum is zero modulo $k$, then there exists a rearranged subsequence $\{b_i\}_{i=1}^n$ of $S$ such that $\sum_{i=1}^n w_ib_i=0$. This extends the Erdős-Ginzburg-Ziv Theorem, which is the case when $m=n$ and $w_i=1$ for all $i$, and confirms a conjecture of Y. Caro. Furthermore, we in part verify a related conjecture of Y. Hamidoune, by showing that if $S$ has an $n$-set partition $A=A_1,\ldots,A_n$ such that $|w_iA_i|=|A_i|$ for all $i$, then there exists a nontrivial subgroup $H$ of $G$ and an $n$-set partition $A'=A'_1,\ldots,A'_n$ of $S$ such that $H\subseteq\sum_{i=1}^n w_iA'_i$ and $|w_iA'_i|=|A'_i|$ for all $i$, where $w_iA_i=\{w_ia_i \mid a_i\in A_i\}$.

## 1 Introduction

Let $(G, +, 0)$ be an abelian group. If $A, B\subseteq G$, then their *sumset*, $A+B$, is the set of all possible pairwise sums, i.e. $\{a+b \mid a\in A, b\in B\}$. A set $A\subseteq G$ is $H_a$-*periodic*, if it is the union of $H_a$-cosets for some subgroup $H_a$ of $G$ (note this definition of periodic differs slightly from the usual definition by allowing $H_a$ to be trivial). A set which is maximally $H_a$-periodic, with $H_a$ the trivial group, is *aperiodic*, and otherwise we refer to $A$ as *nontrivially periodic.* We say that $A$ is *maximally* $H_a$-periodic, if $A$ is $H_a$-periodic, and $H_a$ is the maximal subgroup for which $A$ is periodic; in this case, $H_a = \{x \in G \mid x + A = A\}$, and $H_a$ is sometimes referred to as the *stabilizer* of $A$. Note that if $A$ is $H_{ka}$-periodic, then $A + B$ is also $H_{ka}$-periodic, whence $H_{ka}$ must be a subgroup of the group $H_a$ for which $A + B$ is maximally $H_a$-periodic. For a subgroup $H$, an $H$-*hole* of $A$ (where the subgroup $H$ is usually understood) is an element $\alpha \in (A + H) \setminus A$. The *exponent* of $G$ for $G$ finite is the minimal integer $k$ such that $kg = 0$ for all $g \in G$. We will regard $G$ as a $\mathbb{Z}$-module, and will use $\phi_a : G \to G/H_a$ to denote the natural homomorphism. For $w \in \mathbb{Z}$ and $A \subseteq G$, we let $wA = \{wa_i \mid a_i \in A\}$. If $S$ is a sequence of elements from $G$, then an *n-set partition* of $S$ is a collection of $n$ nonempty subsequences of $S$, pairwise disjoint as sequences, such that every term of $S$ belongs to exactly one of the subsequences, and the terms in each subsequence are all distinct. Thus such subsequences can be considered as sets. We say that $S$ is *zero-sum* if the sum of the terms of $S$ is zero. A *rearranged subsequence* of $S$ is a sequence that under some permutation of terms is a subsequence of $S$.

In 1961, Erdős, Ginzburg and Ziv proved the following theorem, which can be thought of as a generalization of the pigeonhole principle for $m$ pigeons and two holes [12] [1] [32].

**Erdős-Ginzburg-Ziv Theorem (EGZ).** *If $S$ is a sequence of $2m - 1$ elements from an abelian group $G$ of order $m$, then $S$ contains an $m$-term zero-sum subsequence.*

Their theorem spurred the growth of the now developing field of zero-sum Ramsey theory [4] [5] [10] [15] [22] [35], and has been the subject of numerous and varied generalizations [3] [6] [7] [8] [9] [13] [14] [16] [17] [18] [20] [25] [33] [34]. In the early 1990's, N. Alon proved the following conjecture of Y. Caro in the case $n = m$ with $m$ prime.

**Conjecture 1.1.** *Let $n \geq 2$ and $m$ be integers. Let $W = \{w_i\}_{i=1}^{n}$ be a sequence of integers such that $\sum_{i=1}^{n} w_i \equiv 0 \mod m$. Let $S = \{b_i\}_{i=1}^{m+n-1}$ be another sequence of integers. Then there exists a rearranged subsequence $\{b_{j_i}\}_{i=1}^{n}$ of $S$ such that $\sum_{i=1}^{n} w_i b_{j_i} \equiv 0 \mod m$.*

After communicating with A. Bialostocki and Y. Caro, the proof was soon extended to arbitrary $n$ and $m$ prime, were the status of the problem remained. The conjecture was included a few years later in a survey of Y. Caro on problems in zero-sum combinatorics [10], where a reference was made to the (unpublished) proof of the prime case [2]. Soon after, Y. Hamidoune published a pair of papers where, in the first he proved that an equivalent form of Conjecture 1.1 holds (in a more general abelian group setting) provided each $w_i$ for $i \in \{1, \ldots, n-1\}$ was relatively prime to $m$ [26], and in the second he introduced the following conjecture which he verified for $n = m$ [27].

**Conjecture 1.2.** *Let $n \geq 2$ and $m$ be integers. Let $S$ be a sequence of $m+n-1$ elements from a finite abelian group $G$ of order $m$, and let $W = \{w_i\}_{i=1}^n$ be a sequence of integers whose sum is zero modulo $m$. If the multiplicity of every term of $S$ is at most $n$, and if each $w_i$ for $i \leq n-1$ is relatively prime to $m$, then there is a nontrivial subgroup $H$ of $G$ such that for every $h \in H$ there is a rearranged subsequence $\{b_{h_i}\}_{i=1}^n$ of $S$ with $\sum_{i=1}^{n} w_i b_{h_i} = h$.*

No further progress was made on either conjecture until 2003, when W. Gao and X. Jin established Conjecture 1.1 in the case of $m = p^2$, for $p$ prime [19].

The main result of this paper is the following theorem, which completely affirms Conjecture 1.1 for all $m$ and $W$.

**Theorem 1.1.** *Let $m$, $k$ and $n \geq 2$ be positive integers. If $S$ is a sequence of $m+n-1$ elements from a nontrivial abelian group $G$ of order $m$ and exponent $k$, and if $W = \{w_i\}_{i=1}^n$ is a sequence of integers whose sum is zero modulo $k$, then there exists a rearranged subsequence $\{b_i\}_{i=1}^n$ of $S$ such that $\sum_{i=1}^{n} w_i b_i = 0$. Furthermore, if $S$ has an $n$-set partition $A = A_1, \ldots, A_n$ such that $|w_i A_i| = |A_i|$ for all $i$, then there exists a nontrivial subgroup $H$ of $G$ and an $n$-set partition $A' = A'_1, \ldots, A'_n$ of $S$ with $H \subseteq \sum_{i=1}^{n} w_i A'_i$ and $|w_i A'_i| = |A'_i|$ for all $i$.*

We note that the example $W = (\underbrace{1, \ldots, 1}_{m-2}, 0, 2)$ and $S = (-1, \underbrace{0, \ldots, 0}_{m-1}, \underbrace{1, \ldots, 1}_{m-1})$ with $G = \mathbb{Z}_m$ cyclic shows that in the above theorem we cannot require $\{b_i\}_{i=1}^n$ to be an actual (including order) subsequence of $S$. Also, it is easily seen that a sequence $S$ has an $n$-set partition if and only if its length $|S|$ is at least $n$ and every term of $S$ has multiplicity at most $n$. Hence, since $|w_iA_i| = |A_i|$ for $w_i$ relatively prime to $k$ (and since both conditions (b) and (c) to be stated at the end of the sentence imply there exists an $n$-set partition of $S$ with at least one set $A_i$ of cardinality one), it follows that Theorem 1.1 implies Conjecture 1.2 provided any one of the following conditions also holds: (a) $w_n$ is relatively prime to $m$, or (b) $n \geq m$, or (c) every term of $S$ has multiplicity at most $n-1$.

We conclude the paper with some remarks concerning two weighted analogs of the Cauchy-Davenport Theorem [11] that generalize Theorem 1.1 provided every $w_i$ is relatively prime to $k$.

## 2 Weighted EGZ

We begin by stating the classical theorem of Kneser for sumsets [30] [28] [31] [29] [32] [24]. The case with $m$ prime is known as the Cauchy-Davenport Theorem [11], and was the original tool used in the proof of Conjecture 1.1 for $m$ prime.

**Kneser's Theorem.** *Let $G$ be an abelian group, and let $A_1, A_2, \ldots, A_n$ be a collection of finite, nonempty subsets of $G$. If $\sum_{i=1}^{n} A_i$ is maximally $H_a$-periodic, then*

$$\left|\sum_{i=1}^{n} \phi_a(A_i)\right| \geq \sum_{i=1}^{n} |\phi_a(A_i)| - n + 1.$$

Note that if $A$ is maximally $H_a$-periodic, then $\phi_a(A)$ is aperiodic. Also, observe that if $A+B$ is maximally $H_a$-periodic and $\rho = |A+H_a|-|A|+|B+H_a|-|B|$ is the number of holes in $A$ and $B$, then Kneser's Theorem implies $|A+B| \geq |A|+|B|-|H_a|+\rho$. Consequently, if either $A$ or $B$ contains a unique element from some $H_a$-coset, then $|A+B| \geq |A|+|B|-1$. More generally, if $\rho = \sum_{i=1}^{n}(|A_i+H_a|-|A_i|)$ is the total number of holes in the $A_i$, then $|\sum_{i=1}^{n} A_i| \geq \sum_{i=1}^{n}|A_i| - (n-1)|H_a| + \rho$.

We will also need the following theorem, which for abelian groups is an easy consequence of Kneser's Theorem (Theorem 3.2 and Lemma 3.3 in [29]).

**Theorem 2.1.** *Let $A$ and $B$ be finite, nonempty subsets of a group with $|B| \geq 2$, and let $b \in B$. If $A + B \neq A + (B \setminus \{b\})$, then $|A+B| \geq |A|+|B|-1$.*

We can now begin the proof of Theorem 1.1, which follows ideas from the proof of a recent composite analog of the Cauchy-Davenport Theorem [23]. In the proof, we will essentially be considering an $n$-set partition $A = A_1, \ldots, A_n$ of $S$ that iteratively maximizes $\sum_{i=1}^{n}|w_iA_i|$, $|\sum_{i=1}^{n}w_iA_i|$, and $\sum_{i=1}^{n}|\phi_a(w_iA_i)|$, where $\sum_{i=1}^{n}w_iA_i$ is maximally $H_a$-periodic. With the help of Kneser's Theorem, we will be able to show that we can remove some term $b$ of $S$ from the set partition $A$ leaving the third maximized quantity unaffected. If the second maximized quantity is also preserved, then this will allow us to place the term $b$ back into the $n$-set partition in such a way as to preserve the first quantity and increase one of the later two quantities, a contradiction, unless the term $\phi_a(b)$ is contained in every set $w_iA_i$, in which case $H_a = \sum_{i=1}^{n}w_ib + H_a \subseteq \sum_{i=1}^{n}w_iA_i$ will follow from Kneser's Theorem, completing the proof. On the other hand, if removing the term $b$ from its set $w_jA_j$ would destroy the second maximized quantity, then we will use Theorem 2.1 to show that the set $w_jA_j$ locally adds lots of elements to the sumset $\sum_{i=1}^{n}w_iA_i$. An extremal argument will then be used to show that either there must be a term of $S$ that can be removed from $A$ while preserving both the later two maximized quantities, or else there will be many sets $w_iA_i$ which locally add lots of elements to $\sum_{i=1}^{n}w_iA_i$, enough so that we can conclude that the sumset $\sum_{i=1}^{n}w_iA_i$ has large enough cardinality globally to represent every element of $G$.

*Proof.* If there is a term $x$ of $S$ whose multiplicity is at least $n + 1$, then $S$ cannot have an $n$-set partition and Theorem 1.1 follows by choosing $b_i = x$ for all $i$. Hence we may assume each term of $S$ has multiplicity at most $n$, whence it easily follows that there exists an $n$-set partition $A = A_1, \ldots, A_n$ of $S$. Choose $A$ such that $\sum_{i=1}^{n}|w_iA_i|$ is maximal.

Suppose $|w_jA_j| < |A_j|$ for some index $j$ (so that the conditions from the furthermore part of Theorem 1.1 do not hold), and let $b, b' \in A_j$ with $w_jb = w_jb'$ and $b \neq b'$. If there exists an index $r$ such that $w_rb \notin w_rA_r$, then the $n$-set partition defined by $A'_j = A_j \setminus \{b\}$, $A'_r = A_r \cup \{b\}$ and $A'_i = A_i$ for $i \neq j, r$, contradicts the maximality of $\sum_{i=1}^{n}|w_iA_i|$. Thus we may assume $w_ib \in w_iA_i$ for all $i$. Hence, since $\sum_{i=1}^{n}w_i \equiv 0 \mod k$, it follows that $0 = \sum_{i=1}^{n}w_ib \in \sum_{i=1}^{n}w_iA_i$, and the proof is complete by an appropriate selection of a term from each $A_i$. So we may assume $|w_iA_i| = |A_i|$ for all $i$. Furthermore, assume $A$ is chosen such that $|\sum_{i=1}^{n} w_iA_i|$ is maximal subject to $|w_iA_i| = |A_i|$ for all $i$.

If $|\sum_{i=1}^{n} w_iA_i| \geq m$, then the proof is complete with $H = G$. Hence, since $|S| = m+n-1$ and since $|w_iA_i| = |A_i|$, it follows that we may assume that

$$|\sum_{i=1}^{n} w_iA_i| < \sum_{i=1}^{n} |w_iA_i| - n + 1, \qquad (1)$$

whence from Kneser's Theorem it follows that $\sum_{i=1}^{n} w_iA_i \stackrel{def}{=} X$ is maximally $H_a$-periodic for some proper, nontrivial subgroup $H_a$ of $G$. Assume that $A$ was chosen, from among all $n$-set partitions $A' = A'_1, \ldots, A'_n$ of $S$ that satisfy $|w_iA'_i| = |A'_i|$ and $\sum_{i=1}^{n} w_iA'_i = X$, such that $\sum_{i=1}^{n} |\phi_a(w_iA_i)|$ is maximal.

If every set $w_iA_i$ with $i \geq 2$ contains an element which is the unique element from its $H_a$-coset in $w_iA_i$, then there are at least $(n-1)(|H_a| - 1)$ holes among the sets $w_iA_i$, whence Kneser's Theorem implies that $|\sum_{i=1}^{n} w_iA_i| \geq \sum_{i=1}^{n} |w_iA_i| - (n-1)|H_a| + (n-1)(|H_a| - 1) = \sum_{i=1}^{n} |A_i| - n + 1$, contradicting (1). Therefore we may assume $|\phi_a(w_jA_j)| < |w_jA_j|$ for some index $j \geq 2$.

Let $j \geq 2$ be an index such that $|\phi_a(w_jA_j)| < |w_jA_j|$. Suppose that

$$\sum_{i=1}^{j} w_iA_i = \sum_{i=1}^{j-1} w_iA_i + w_j(A_j \setminus \{b\}), \qquad (2)$$

for some $b \in A_j$ such that $\phi_a(w_jA_j) = \phi_a(w_j(A_j \setminus \{b\}))$. Hence, if there exists an index $r$ such that $\phi_a(w_rb) \notin \phi_a(w_rA_r)$, then the $n$-set partition defined by $A'_j = A_j \setminus \{b\}$, $A'_r = A_r \cup \{b\}$ and $A'_i = A_i$ for $i \neq j$, $r$, contradicts the maximality of either $|\sum_{i=1}^{n} w_iA_i|$ or $\sum_{i=1}^{n} |\phi_a(w_iA_i)|$. So we may assume $\phi_a(w_ib) \in \phi_a(w_iA_i)$ for all $i$. Hence, since $\sum_{i=1}^{n} w_i \equiv 0 \mod k$, it follows that $0 = \sum_{i=1}^{n} \phi_a(w_ib) \in \sum_{i=1}^{n} \phi_a(w_iA_i)$. Thus, since $\sum_{i=1}^{n} w_iA_i$ is $H_a$-periodic, it follows that $H_a \subseteq \sum_{i=1}^{n} w_iA_i$, and the proof is complete with $H = H_a$. So we may assume that (2) does not hold, whence in view of Theorem 2.1 it follows that

$$|\sum_{i=1}^{j} w_iA_i| \geq |\sum_{i=1}^{j-1} w_iA_i| + |w_jA_j| - 1. \qquad (3)$$

Let $l$, where $2 \leq l \leq n$, be the minimal integer, allowing re-indexing of the $w_iA_i$, such that

$$|\sum_{i=1}^{j} w_iA_i| \geq |\sum_{i=1}^{j-1} w_iA_i| + |w_jA_j| - 1, \tag{4}$$

for all $j \geq l$. From the conclusions of the last two paragraphs, and since by re-indexing we may assume $j = n$ in the previous paragraph, it follows that $l$ exists. Observe that

$$|\sum_{i=1}^{l-1} w_iA_i| < \sum_{i=1}^{l-1} |w_iA_i| - (l-1) + 1, \tag{5}$$

since otherwise applying (4) iteratively yields $|\sum_{i=1}^{n} w_iA_i| \geq \sum_{i=1}^{n} |w_iA_i| - n - 1$, contradicting (1). Hence from Kneser's Theorem and the maximality of $H_a$, it follows that $\sum_{i=1}^{l-1} w_iA_i$ is maximally $H_{ka}$-periodic for some nontrivial subgroup $H_{ka} \leq H_a$. Note that (5) can only hold provided $l - 1 \geq 2$. Furthermore, if every set $w_iA_i$ with $2 \leq i \leq l - 1$ contains an element which is the unique element from its $H_{ka}$-coset in $w_iA_i$, then there will be at least $(l-2)(|H_{ka}|-1)$ holes among the sets $w_iA_i$ with $i \leq l-1$, whence Kneser's Theorem implies that $|\sum_{i=1}^{l-1} w_iA_i| \geq \sum_{i=1}^{l-1} |w_iA_i| - (l-2)|H_{ka}| + (l-2)(|H_{ka}|-1) = \sum_{i=1}^{l-1} |A_i| - (l-1) + 1$, contradicting (5). Therefore there must exist a set $A_j$ with $2 \leq j \leq l - 1$, such that $w_jA_j$ does not contain an element which is the unique element from its $H_{ka}$-coset in $w_jA_j$. Hence, since $H_{ka} \leq H_a$, it follows that $|\phi_a(w_jA_j)| < |w_jA_j|$ for some index $j$ with $2 \leq j \leq l - 1$. Thus, since by re-indexing we may assume $j = l - 1$, it follows that (3) holds with $j = l - 1$, which in view of (4) contradicts the minimality of $l$. □

Theorem 1.1 has a much more potent generalization, provided every $w_i$ is relatively prime to $k$. The following theorem, which implies Theorem 1.1 when every $w_i$ is relatively prime to $k$, was proved in the case $w_i = 1$ for all $i$ in [23]. However, noting for $w_i$ relatively prime to $k$ that $w_ib \in w_iA_i$ if and only if $b \in A_i$, and allowing for re-indexing of the weights $\{w_i\}_{i=1}^{n}$, it follows that the proof given in [23] goes through in the more general weighted case with no further modifications other than to insert the weights $w_i$ at appropriate points in the proof.

**Theorem 2.2.** *Let $S$ be a finite sequence of elements from an abelian group $G$ whose torsion subgroup has exponent $k$, let $W = \{w_i\}_{i=1}^{n}$ be a sequence of integers relatively* *prime to $k$, let $A = A_1, \ldots, A_n$ be an $n$-set partition of $S$, and let $a_i \in A_i$ for $i \in \{1, \ldots, n\}$. Then there exists an $n$-set partition $A' = A'_1, \ldots, A'_n$ of $S$ such that $\sum_{i=1}^{n} w_i A'_i$ is $H_a$-periodic, $\sum_{i=1}^{n} w_i A_i \subseteq \sum_{i=1}^{n} w_i A'_i$, $a_i \in A'_i$ for $i \in \{1, \ldots, n\}$, and*

$$\left|\sum_{i=1}^{n} w_i A'_i\right| \geq (E(A', H_a) + (N(A', H_a) - 1)n + 1)\, |H_a|,$$

*where $N(A', H_a) = \frac{1}{|H_a|} |\bigcap_{i=1}^{n} (A'_i + H_a)|$ and $E(A', H_a) = \sum_{j=1}^{n} (|A'_j| - |A'_j \cap \bigcap_{i=1}^{n} (A'_i + H_a)|)$.*

Finally, we remark that the proof when $w_i = 1$ for all $i$ of the following corollary to Theorem 2.2 likewise goes through in the more general weighted case (Theorem 2.7 in [21] and Corollary 1 in [23]).

**Theorem 2.3.** *Let $S$ be a sequence of elements from an abelian group $G$ of order $m$ and exponent $k$ with an $n$-set partition $P = P_1, \ldots, P_n$, let $W = \{w_i\}_{i=1}^{n}$ be a sequence of integers relatively prime to $k$, and let $p$ be the smallest prime divisor of $m$. Then either:*

*$(i)$ there exists an $n$-set partition $A = A_1, A_2, \ldots, A_n$ of $S$ such that:*

$$|\sum_{i=1}^{n} w_i A_i| \geq \min\{m,\, (n+1)p,\, |S| - n + 1\}\,;$$

*furthermore, if $n' \geq \frac{m}{p} - 1$ is an integer such that $P$ has at least $n - n'$ cardinality one sets and if $|S| \geq n + \frac{m}{p} + p - 3$, then we may assume there are at least $n - n'$ cardinality one sets in $A$, or*

*$(ii)$ $(a)$ there exists $\alpha \in G$ and a nontrivial proper subgroup $H_a$ of index $a$ such that all but at most $a - 2$ terms of $S$ are from the coset $\alpha + H_a$; and $(b)$ there exists an $n$-set partition $A_1, A_2, \ldots, A_n$ of the subsequence of $S$ consisting of terms from $\alpha + H_a$ such that $\sum_{i=1}^{n} w_i A_i = \sum_{i=1}^{n} w_i \alpha + H_a$.*

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