A Five Color Zero-Sum Generalization

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Abstract

Let $g_{zs}(m,2k)$ ($g_{zs}(m,2k+1)$) be the minimal integer such that for any coloring Δ of the integers from $1,\ldots,g_{zs}(m,2k)$ by $\biguplus_{i=1}^{k}\mathbb{Z}_m^i$ (the integers from $1,\ldots,g_{zs}(m,2k+1)$ by $\biguplus_{i=1}^{k}\mathbb{Z}_m^i\cup\{\infty\}$) there exist integers

$$x_1<\cdots<x_m<y_1<\cdots<y_m$$

such that

1. there exists j_x such that $\Delta(x_i)\in\mathbb{Z}_m^{j_x}$ for each i and $\sum_{i=1}^{m}\Delta(x_i)=0\mod m$ (or $\Delta(x_i)=\infty$ for each i);
2. there exists j_y such that $\Delta(y_i)\in\mathbb{Z}_m^{j_y}$ for each i and $\sum_{i=1}^{m}\Delta(y_i)=0\mod m$ (or $\Delta(y_i)=\infty$ for each i); and
3. $2(x_m-x_1)\le y_m-x_1$.

In this note we show $g_{zs}(m,2)=5m-4$ for $m\ge 2$, $g_{zs}(m,3)=7m+\left\lfloor\frac{m}{2}\right\rfloor-6$ for $m\ge 4$, $g_{zs}(m,4)=10m-9$ for $m\ge 3$, and $g_{zs}(m,5)=13m-2$ for $m\ge 2$.

1 Introduction

Denote by $[a,b]$ the set of integers x such that $a\le x\le b$. For a set S, an S-coloring of $[a,b]$ is a mapping $\Delta:[a,b]\to S$. If $S=\{1,\ldots,r\}$, we say Δ is

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an r-coloring. For A, $B \subseteq G$, where G is an abelian group, then their *sumset*, $A + B$, is the set of all possible pairwise sums, i.e. $\{a + b \mid a \in A,\, b \in B\}$. The following is the Erdős-Ginzburg-Ziv Theorem (EGZ), [9] [8].

Erdős-Ginzburg-Ziv Theorem (EGZ). *Any sequence of at least $2m - 1$ elements of $\mathbb{Z}_m$ contains a subsequence of m elements whose sum is zero modulo m.*

Several theorems of Ramsey-type have been generalized by considering Z_m-colorings and zero-sum configurations rather than 2-colorings and monochromatic configurations. Such theorems for which the corresponding Ramsey number for 2-colorings equals the corresponding Ramsey number for the Z_m-colorings are said to zero-sum generalize in the sense of the EGZ Theorem. Best known of these results is the zero-sum-tree theorem [4] [19], and other results concerning graphs and hypergraphs can be found in [10] and [1].

Ramsey-type problems dealing with colorings of the natural numbers can be classified as one-set problems initiated in [7] and further explored in [3] [6] [15] [16] [17] [18], and two-set problems initiated in [5] and further investigated in [13] [20] [21]. We introduce some definitions towards generalizing two-set problems in the sense of the EGZ Theorem.

Let the set $\biguplus_{i=1}^{k} \mathbb{Z}_m^i$ denote the pairwise disjoint union of k copies of the set of elements $\mathbb{Z}_m$ and let ∞ denote a symbol such that $\infty \notin \biguplus_{i=1}^{k} \mathbb{Z}_m^i$. Just as the EGZ Theorem generalizes the pigeonhole principle for 2 boxes and m pigeons, the following observation generalizes Proposition 1.1 to an arbitrary number of boxes.

Observation 1.1. *Let $m \geq 2$ and $r = 2k$ ($r = 2k + 1$) be positive integers. Any sequence of at least $r(m-1)+1$ elements from $\biguplus_{i=1}^{k} \mathbb{Z}_m^i$ (from $\biguplus_{i=1}^{k} \mathbb{Z}_m^i \cup \{\infty\}$) contains a subsequence of m elements from some $\mathbb{Z}_m^i$ whose sum is zero modulo m (or a subsequence of m ∞ elements).*

For a positive integer r and a system of inequalities L in $2m$ variables, let $R(L; r)$ denote the minimal integer N such that every r-coloring of $[1, N]$ contains two sets S_1 and S_2, each being monochromatic and of cardinality m, such that $S_1 \cup S_2$ forms a solution to L. In a similar way, if $r = 2k$ ($r = 2k + 1$) let $R_{zs}(L; r)$ denote the minimum integer N such that every $\biguplus_{i=1}^{k} \mathbb{Z}_m^i$-coloring ($\biguplus_{i=1}^{k} \mathbb{Z}_m^i \cup \{\infty\}$-coloring) of $[1, N]$ contains two sets S_1 and S_2, each being zero-sum in $\mathbb{Z}_m^i$ (or ∞-monochromatic) and of cardinality m, such that $S_1 \cup S_2$ forms a solution to L.

It is easy to see that $R(L;r) \leq R_{zs}(L;r)$. If equality holds for a given r, we say the system L admits an EGZ generalization for r colors. Using the definitions above, it was proved in [5] that $R(\bar{L};2) = 5m-3$ and $R(\bar{L};3) = 9m-7$ where

$$\begin{aligned}\bar{L} := x_1 < x_2 < \cdots < x_m < y_1 < y_2 < \cdots < y_m;\\ x_m - x_1 \leq y_m - y_1.\end{aligned}$$

Furthermore, they proved $\bar{L}$ admits an EGZ generalization for 2 and 3 colors.

Recently in a sequence of three papers [13], [12], [11] the first author showed $R(\bar{L};4) = 12m-9$ and that $\bar{L}$ admits an EGZ generalization for 4 colors. In achieving this result, a new tool was developed in [11]. We state in Proposition 2.1 a particular case equivalent to a result from [14]. It seems that the determination of $R(\bar{L};5)$ would not be easy or short. At present the authors are not aware of any nontrivial two set EGZ generalizations for 5 colors.

The motivation for this paper is twofold. First, we wished to find a system $\mathcal{L}$ that admits an EGZ generalization for 5 colors. Second, we wanted to test the conjecture below from [2].

Conjecture 1.3 *Let k and m be positive integers. If L_1 and L_2 are two systems of inequalities in $2m$ variables such that every positive integer solution of L_1 is a solution of L_2, and if L_2 admits an EGZ generalization in r colors, then L_1 admits an EGZ generalization in r colors.*

Toward these ends we have chosen to look at the system $\mathcal{L}$ first investigated by the second author in [20] defined by

$$\begin{aligned}\mathcal{L} := x_1 < x_2 < \cdots < x_m < y_1 < y_2 < \cdots < y_m;\\ y_m - x_1 \geq 2(x_m - x_1).\end{aligned}$$

In Section 2 we state some preliminary definitions and tools, and in Section 3 we determine $R_{zs}(\mathcal{L};r)$ for $r \in [2,5]$. In conjunction with the results from [20], these results show $\mathcal{L}$ admits EGZ generalizations for these values of r.

2 Preliminaries

Along with the EGZ theorem we shall need the following result, an easy consequence of the EGZ Theorem and [11] or [14].

Proposition 2.1. *If $H = (h_1, \ldots, h_k)$ is a sequence of at least $2m-1$ elements from $\mathbb{Z}_m$, then one of the following holds:*

1. *there exists a divisor a of m satisfying $1 < a < m$ such that any sequence of $m + \frac{m}{a} + a - 3 \leq \lfloor \frac{3}{2}m \rfloor - 1$ elements from H contains a subsequence of m elements whose sum is zero modulo m*
2. *there exists a partition of H into sets $\{A_i\}_{i=1}^{k-m+1}$ with $|\sum_{i=1}^{k-m+1} A_i| = m$; or*
3. *there exists $j \in \mathbb{Z}_m$ such that $h_i = j$ for all but at most $m-2$ terms $h_i \in H$.*

An m-set, denoted $Z = (z_1, \ldots, z_m)$, is a sequence of m distinct positive integers such that $z_1 < \cdots < z_m$. For a pair of m-sets X and Y, we write $X \prec Y$ if $x_m < y_1$. For $Z = (z_1, \ldots, z_m)$ we also adopt the following notation:
(i) $int_i(Z) = z_i$ for $i \leq m$;
(ii) $first_k(Z) = \{z_1, \ldots, z_{\min\{k,m\}}\}$;
(iii) $first(Z) = z_1$;
(iv) $last_k(Z) = \{z_{\max\{1,n-k\}}, \ldots, x_m\}$; and
(v) $last(Z) = z_m$.

For matters of notation and consistency with [20], we shall denote $R_{zs}(\mathcal{L}, r)$ by $g_{zs}(m, r)$. To facilitate our evaluation of $g_{zs}(m, r)$, we make the following observation.

Observation 2.2. *Let positive integer $r = 2k$ (integer $r = 2k+1$) be given, and let $\Delta : [1, n] \to \biguplus_{i=1}^{k} \mathbb{Z}_m^i$ ($\Delta : [1, n] \to \biguplus_{i=1}^{k} \mathbb{Z}_m^i \cup \{\infty\}$) be given. If there exists a zero-sum (zero-sum or monochromatic) m-set $Y \subset [r(m-1)+2, n]$ such that $y_m \geq 2r(m-1)+1$, then the system $\mathcal{L}$ is satisfied.*

Proof. By Observation 1.2 there is some zero-sum or monochromatic m-set $X \subset [1, r(m-1)+1]$. If a zero-sum or monochromatic m-set $Y \subset [r(m-1)+2, n]$ exists, then $X \prec Y$. If $y_m \geq 2r(m-1)+1$ we have $y_m - x_1 \geq 2r(m-1)+1-x_1 \geq 2(r(m-1)+1-x_1) \geq 2(x_m - x_1)$. □

3 Evaluation of $g_{zs}(m, r)$ for $r \in [2, 5]$

The determination of $g_{zs}(m, 2)$ is a simple application of the EGZ Theorem.

Theorem 3.1. *If $m \geq 2$ is an integer, then $g_{zs}(m, 2) = 5m - 4$.*

Proof. That $g_{zs}(m, 2) \geq 5m - 4$ follows from $g(m, 2) = 5m - 4$ as found in [20] and the trivial fact that $g_{zs}(m, 2) \geq g(m, 2)$.

By Observation 2.2 it is sufficient to find a zero-sum m-set $Y \subset [2m, 5m-4]$ with $y_m \geq 4m - 3$. Let $P = [3m - 2, 5m - 4]$. Since $|P| = 2m - 1$ there exists some zero-sum m-set $Y \subset P$. Since $|P \cap [3m - 2, 4m - 4]| = m - 1$ it follows that $y_m \geq 4m - 3$. □

The determination of $g_{zs}(m, 3)$ will require the use of Proposition 2.1 and the following two lemmas, proofs for which can be found in [20].

Lemma 3.2. *Let $m \geq 4$ be an integer, and let $\Delta : [1, 3m - 4] \to \mathbb{Z}_m \cup \{\infty\}$ be a coloring. If $|\Delta^{-1}(\infty)| \geq 3m - \lceil \frac{m}{2} \rceil - 2$, then there exist monochromatic m-sets $X \prec Y$ such that $y_m - x_1 \geq 2(x_m - x_1)$.*

Lemma 3.3. *Let $m \geq 4$ be an integer. If $\Delta : [3m-1, 7m+\lfloor \frac{m}{2} \rfloor - 6] \to [1, 3]$ is a given coloring, then either*

1. *there exists a monochromatic m-set Y such that $y_m \geq 6m - 5$ or*
2. *there exist monochromatic m-sets $W \prec Y$ such that $y_m - w_1 \geq 2(w_m - w_1)$.*

Theorem 3.4. *If $m \geq 4$ is an integer, then $g_{zs}(m, 3) = 7m + \lfloor \frac{m}{2} \rfloor - 6$.*

Proof. That $g_{zs}(m, 3) \geq 7m + \lfloor \frac{m}{2} \rfloor - 6$ follows from $g(m, 3) = 7m + \lfloor \frac{m}{2} \rfloor - 6$ as found in [20] and the trivial fact that $g_{zs}(m, 3) \geq g(m, 3)$.

Next we show that $g_{zs}(m, 3) \leq 7m+\lfloor \frac{m}{2} \rfloor -6$. Let $\Delta : [1, 7m+\lfloor \frac{m}{2} \rfloor -6] \to \mathbb{Z}_m \cup \{\infty\}$ be an arbitrary coloring. By Observation 2.2 it is sufficient to find a zero-sum or monochromatic m-set $Y \subset [3m - 1, 7m + \lfloor \frac{m}{2} \rfloor - 6]$ with $y_m \geq 6m - 5$.

For convenience we let

$$P = \Delta^{-1}(\mathbb{Z}_m) \cap [3m - 1, 7m + \left\lfloor \frac{m}{2} \right\rfloor - 6].$$

To complete the proof we consider three cases based on $k = |\Delta^{-1}(\infty) \cap [6m-5, 7m + \lfloor \frac{m}{2} \rfloor - 6]|$. Note $k < m$ holds trivially.

Case 1. Suppose $k = 0$. If $|P| \geq 2m - 1$, then one may find a zero-sum m-set $Y \subset [3m - 1, 7m + \lfloor \frac{m}{2} \rfloor - 6]$ with $y_m \geq 6m - 5$ by selecting $P' \subset P$ such that $|P'| = 2m - 1$ and $|P' \cap [6m - 5, 7m + \lfloor \frac{m}{2} \rfloor - 6]| = m$ and applying Proposition 1.1. Otherwise $|P| < 2m-1$, so that $|\Delta^{-1}(\infty) \cap [3m-1, 6m-6]| \geq$

$3m - \left\lceil \frac{m}{2} \right\rceil - 2$. Shifting $[3m-1, 6m-6]$ to the interval $[1, 3m-4]$ and applying Lemma 3.2 completes the proof.

Case 2. Suppose $0 < k \leq \left\lfloor \frac{m}{2} \right\rfloor$, so that

$$|P \cap [6m-5, 7m + \left\lfloor \frac{m}{2} \right\rfloor - 6]| = m + \left\lfloor \frac{m}{2} \right\rfloor - k \geq m.$$

If $|\Delta^{-1}(\infty) \cap [3m-1, 6m-6]| \geq m-1$, then we are done. So we may assume that $|\Delta^{-1}(\mathbb{Z}_m) \cap [3m-1, 6m-6]| \geq 2m-2$. Selecting $P' \subset P$ such that $|P'| = 2m-1$ and $|P \cap [6m-5, 7m + \left\lfloor \frac{m}{2} \right\rfloor - 6]| = m$ completes the case.

Case 3. Suppose $\left\lfloor \frac{m}{2} \right\rfloor < k < m$. We may assume that $|\Delta^{-1}(\infty) \cap [3m-1, 6m-6]| < m-k$, since otherwise the proof is complete by taking $Y = last_m(\Delta^{-1}(\infty) \cap [3m-1, 7m + \left\lfloor \frac{m}{2} \right\rfloor - 6])$. Hence, $|P| > 2m-1$. We complete this case by considering three subcases corresponding to the three possible conclusions of Proposition 2.1.

Subcase 3a. Suppose any $\lfloor \frac{3}{2}m \rfloor$ elements of P contain a zero-sum m-set. Since $|P \cap [6m-5, 7m + \left\lfloor \frac{m}{2} \right\rfloor - 6]| \geq \left\lfloor \frac{m}{2} \right\rfloor + 1$ we may select $P' \subset P$ with $\lfloor \frac{3}{2}m \rfloor$ elements such that $|P' \cap [6m-5, 7m + \left\lfloor \frac{m}{2} \right\rfloor - 6]| \geq \left\lfloor \frac{m}{2} \right\rfloor + 1$. By assumption P' contains a zero-sum m-set Y. Since $|P' \cap [3m-1, 6m-6]| < m$ it follows that $y_m \geq 6m-5$.

Subcase 3b. Suppose there exists a partition of $P \setminus \{last(P)\}$ into sets $\{A_i\}_{i=1}^{|P|-m}$ with

$$|\sum_{i=1}^{|P|-m} A_i| = m.$$

Hence, since $k < m$, and noting that there can be at most $m-1$ sets A_i in P with $|A_i| > 1$, it follows that there exists a zero-sum m-set Y with $y_m = last(P) \geq 6m-5$.

Subcase 3c. If neither Subcase 3a nor 3b apply, then applying Proposition 2.1 to $P \setminus \{last(P)\}$ it follows that $\Delta(p) = j \in \mathbb{Z}_m$ for all but at most $m-2+1$ elements $p \in P$; for convenience, let $H = \{p \in P | \Delta(p) \neq j\}$. Induce a coloring $\Delta_e : [3m-1, 7m + \left\lfloor \frac{m}{2} \right\rfloor - 6] \to [1, 3]$ defined by

$$\Delta_e(x) = \begin{cases} 1, & \text{for } x \in P \setminus H \\ 2, & \text{for } x \in H \\ 3, & \text{for } \Delta(x) = \infty \end{cases}$$

Note that any monochromatic m-set W in Δ_e is either zero-sum or monochromatic in Δ since $|\Delta^{-1}(2)| \leq m-1$. Hence, the result follows by Lemma 3.3. □

We consider the evaluation of $g_{zs}(m,4)$. Towards that end we use the following lemma, the proof for which may be found in [20].

Lemma 3.5. *Let $m \geq 3$ be an integer. If $\Delta : [4m-2, 10m-9] \to [1,4]$ is a given coloring, then either*

1. *there exists a monochromatic m-set Y such that $y_m \geq 8m-7$ or*
2. *there exist monochromatic m-sets $W \prec Y$ such that $y_m - w_1 \geq 2(w_m - w_1)$.*

Theorem 3.6. *If $m \geq 3$ is an integer, then $g_{zs}(m,4) = 10m-9$.*

Proof. That $g_{zs}(m,4) \geq 10m-9$ follows from $g(m,4) = 10m-9$ as found in [20] and the trivial fact that $g_{zs}(m,4) \geq g(m,4)$.

Next we show that $g_{zs}(m,4) \leq 10m-9$. Let $\Delta : [1, 10m-9] \to \mathbb{Z}_m^1 \uplus \mathbb{Z}_m^2$ be an arbitrary coloring. By Observation 2.2 it is sufficient to find a zero-sum m-set $Y \subset [4m-2, 10m-9]$ with $y_m \geq 8m-7$.

Since $|[8m-7, 10m-9]| = 2m-1$, without loss of generality we may assume $|\Delta^{-1}(\mathbb{Z}_m^2) \cap [8m-7, 10m-9]| = m+k$ where $k \geq 0$. If $|\Delta^{-1}(\mathbb{Z}_m^2) \cap [4m-2, 8m-8]| \geq m-1-k$, then by the EGZ theorem there exists a zero-sum m-set $Y \subset [4m-2, 10m-9]$ with $y_m \geq 8m-7$. So we may assume

$$|\Delta^{-1}(\mathbb{Z}_m^2) \cap [4m-2, 8m-8]| \leq m-2-k. \tag{1}$$

Letting $P = \Delta^{-1}(\mathbb{Z}_m^1) \cap [4m-2, 10m-9]$, we thus have $|P \cap [4m-2, 8m-8]| \geq 3m-3+k$. We finish the proof by considering three cases.

Case 1. Suppose any $\lfloor \frac{3}{2}m \rfloor$ elements of P contain a zero-sum m-set. If such an m-set Y satisfies $y_m \geq 8m-7$ we are done; hence, since $|\Delta^{-1}(\mathbb{Z}_m^1 \cap [8m-7, 10m-9]| \geq 2m-1-(m+k) = m-1-k$, we can assume that any set S of $\lfloor \frac{3}{2}m \rfloor - (m-1-k) = \lfloor \frac{m}{2} \rfloor + k + 1$ elements from $\Delta^{-1}(\mathbb{Z}_m^1) \cap [4m-2, 8m-8]$ contains a zero-sum m-set, since otherwise $S \cup (\Delta^{-1}(\mathbb{Z}_m^1 \cap [8m-7, 10m-9])$ must contain a zero-sum m-set with $y_m \geq 8m-7$. Hence $k \geq \lceil \frac{m}{2} \rceil - 1$ and for $W' = first_{\lfloor \frac{m}{2} \rfloor + k + 1}(\Delta^{-1}(\mathbb{Z}_m^1) \cap [4m-2, 8m-8])$ there exists some m-set $W \subset W'$ that is zero-sum.

By Equation 1 we have

$$t = |\Delta^{-1}(\mathbb{Z}_m^2) \cap [4m-2, w_m]| \leq m-2-k, \tag{2}$$

so that $w_m - w_1 \leq \lfloor \frac{m}{2} \rfloor + k + t - t'$, where $w_1 = (4m-2)+t'$. Hence, if there exists a zero-sum m-set Y such that $W \prec Y$ and

$$y_m \geq 2(\left\lfloor \frac{m}{2} \right\rfloor + k + t) + 4m - 2 \geq 2(\left\lfloor \frac{m}{2} \right\rfloor + k + t - t') + 4m - 2 + t' \geq 2(w_m - w_1) + w_1 \tag{3}$$

we shall be done. Taking $Y' = last_{\lfloor \frac{m}{2} \rfloor + k + 1}(\Delta^{-1}(\mathbb{Z}_m^1) \cap [4m-2, 8m-8])$, we see that there exists an m-set $Y \subset Y'$ which satisfies these requirements as follows. First, it is quickly verified from equation 1 that there are at least $2(\lfloor \frac{m}{2} \rfloor + k + 1)$ many elements from $\Delta^{-1}(\mathbb{Z}_m^1)$ in $[4m-2, 8m-8]$ for every $m \geq 3$. Hence, we have $W' \prec Y'$, from which it follows that $W \prec Y$. Second, we note that by using Equations 1 and 2 it follows that

$$\begin{aligned} y_m \geq 8m - 8 - |Y' \setminus Y| - (|\Delta^{-1}(\mathbb{Z}_m^2) \cap [4m-2, 8m-8]| - t) \geq \\ 8m - 8 - \Big((\left\lfloor \frac{m}{2} \right\rfloor + k + 1) + (m - 2 - k) - m \Big) - ((m - 2 - k) - t) = \\ 6m + \lceil \frac{m}{2} \rceil - 5 + (k + t) \geq m + (m - 2) + (k + t) + (4m - 2) \geq \\ 2(\left\lfloor \frac{m}{2} \right\rfloor + (k + t)) + 4m - 2. \end{aligned}$$

Hence Equation 3 is satisfied.

Case 2. Suppose there exists a partition of $P \setminus \{last(P)\}$ into sets $\{A_i\}_{i=1}^{|P|-m}$ such that

$$|\sum_{i=1}^{|P|-m} A_i| = m.$$

Hence, since $k \leq m-2$ follows from Equation 1, and noting that there can be at most $m-1$ sets A_i in P with $|A_i| > 1$, it follows that there exists a zero-sum m-set $Y \subset [4m-2, 10m-9]$ with $y_m = last(P) \geq 8m-7$.

Case 3. If neither Case 1 nor Case 2 apply, then applying Proposition 2.1 to $P \setminus \{last(P)\}$ it follows that $\Delta(p) = j \in \mathbb{Z}_m^1$ for all but at most $m-1$ elements $p \in P$; for convenience, define $H = \{p \in P | \Delta(p) \neq j\}$. Induce a coloring $\Delta_e : [4m-2, 10m-9] \to [1, 4]$ defined by

$$\Delta_e(x) = \begin{cases} 1, \text{ for } x \in P \setminus H \\ 2, \text{ for } x \in H \\ 3, \text{ for } x \in first_{m-1}(\Delta^{-1}(\mathbb{Z}_m^1) \cap [4m-2, 10m-9]) \\ 4, \text{ for } x = int_i(\Delta^{-1}(\mathbb{Z}_m^1) \cap [4m-2, 10m-9]), m \leq i \leq 2m-2 \end{cases}$$

Note that any monochromatic m-set X in Δ_e is zero-sum in Δ since $|\Delta^{-1}(j)| \leq m-1$ for each $j \neq 1$. Hence, by Lemma 3.5 the proof is complete. □

The evaluation of $g_{zs}(m,5)$ requires two results from the evaluation of $g(m,2)$ and $g(m,5)$ found in [20].

Lemma 3.7. *Let $m \geq 2$ be an integer. If $\Delta : [5m-3, 13m-12] \to [1,5]$ is a given coloring, then either*

1. *there exists a monochromatic m-set Y such that $y_m \geq 10m-9$ or*
2. *there exist monochromatic m-sets $W \prec Y$ such that $y_m - w_1 \geq 2(w_m - w_1)$.*

Lemma 3.8. *Let $m \geq 2$ be an integer. If $\Delta : [5m-3, 10m-10] \to [1,2]$ is a coloring with $|\Delta^{-1}(c)| \leq m-2$ for some $c \in [1,2]$, then there exist monochromatic m-sets $X' \prec Y'$ such that $y'_m - x'_1 \geq 2(x'_m - x'_1)$.*

Theorem 3.9. *If $m \geq 2$ is an integer, then $g_{zs}(m,5) = 13m-12$.*

Proof. That $g_{zs}(m,5) \geq 13m-12$ follows from $g(m,5) = 13m-12$ as found in [20] and the trivial fact that $g_{zs}(m,5) \geq g(m,5)$.

We now show $g_{zs}(m,5) \leq 13m-12$. Let $\Delta : [1, 13m-12] \to \mathbb{Z}_m^1 \uplus \mathbb{Z}_m^2 \cup \{\infty\}$ be an arbitrary coloring. By Observation 2.2 it is sufficient to find a zero-sum or monochromatic m-set $Y \subset [5m-3, 13m-12]$ with $y_m \geq 10m-9$.

The case of $|\Delta^{-1}(\infty) \cap [10m-9, 13m-12]| \geq m$ is trivial, so we may assume otherwise. Hence, it follows that $|\Delta^{-1}(\mathbb{Z}_m^i) \cap [10m-9, 13m-12]| \geq m$ for some $i \in [1,2]$, say $i=2$. Furthermore, if $|\Delta^{-1}(\infty) \cap [10m-9, 13m-12]| = 0$, then either $|\Delta^{-1}(\mathbb{Z}_m^2) \cap [10m-9, 13m-12]| \geq 2m-1$ or $|\Delta^{-1}(\mathbb{Z}_m^1) \cap [10m-9, 13m-12]| \geq m$. In the former case the proof is complete by the EGZ Theorem. In the latter case, from the EGZ Theorem it follows that $|(\Delta^{-1}(\mathbb{Z}_m^1) \cup \Delta^{-1}(\mathbb{Z}_m^2)) \cap [5m-3, 10m-10]| \leq m-2$. Hence we may induce a coloring $\Delta_e : [5m-3, 10m-10] \to [1,2]$ defined by

$$\Delta_e(x) = \begin{cases} 1, & \text{for } \Delta(x) = \infty \\ 2, & \text{for } x \in (\Delta^{-1}(\mathbb{Z}_m^1) \cup \Delta^{-1}(\mathbb{Z}_m^2)) \cap [5m-3, 10m-10]. \end{cases}$$

Since any monochromatic m-set in Δ_e is also monochromatic in Δ, the result follows by Lemma 3.8. So $|\Delta^{-1}(\infty) \cap [10m-9, 13m-12]| > 0$.

Let $k = |(\Delta^{-1}(\infty) \cup \Delta^{-1}(\mathbb{Z}_m^2)) \cap [10m-9, 13m-12]|$. Clearly $k < 3m-2$. Since $|\Delta^{-1}(\infty) \cap [10m-9, 13m-12]| > 0$, and since $|\Delta^{-1}(\mathbb{Z}_m^2) \cap [10m-9, 13m-12]| \geq m$, it follows that

$$|(\Delta^{-1}(\infty) \cup \Delta^{-1}(\mathbb{Z}_m^2)) \cap [5m-3, 10m-10]| < 3m-2-k, \tag{4}$$

since otherwise there exists either a zero-sum or monochromatic m-set Y with $y_m \in [10m-9, 13m-12]$. Likewise, we may assume that any set S of

$$\begin{aligned} 2m-1-|\Delta^{-1}(\mathbb{Z}_m^1)\cap[10m-9,13m-12]| &= 2m-1-(3m-2-k) \\ &= k+1-m \end{aligned}$$

elements from $\Delta^{-1}(\mathbb{Z}_m^1)\cap[5m-3,10m-10]$ contains a zero-sum m-set, since otherwise from the EGZ Theorem it follows that $S\cup(\Delta^{-1}(\mathbb{Z}_m^1)\cap[10m-9,13m-12])$ will contain a zero-sum m-set with $y_m \geq 10m-9$.

Let $W' = first_{k+1-m}(\Delta^{-1}(\mathbb{Z}_m^1)\cap[5m-3,10m-10])$, so that by assumption there exists a zero-sum m-set $W \subset W'$. From Equation 4 we see that

$$t = |(\Delta^{-1}(\infty)\cup\Delta^{-1}(\mathbb{Z}_m^2))\cap[5m-3,w_m]| \leq 3m-3-k \tag{5}$$

so that $w_m - w_1 \leq k-m+t-t'$, where $w_1 = 5m-3+t'$. Hence, if there exists a zero-sum m-set Y such that $W \prec Y$ and such that

$$y_m \geq 3m+2k+2t-3 \geq 2(k-m+t-t')+5m-3+t' \geq 2(w_n-w_1)+w_1, \tag{6}$$

then we shall be done. Taking $Y' = last_{k+1-m}(\Delta^{-1}(\mathbb{Z}_m^1)\cap[5m-3,10m-10])$, we see that there exists an m-set $Y \subset Y'$ which satisfies these requirements as follows. First, from Equation 4 one may verify that there are at least $2(k+1-m)$ many elements from $\Delta^{-1}(\mathbb{Z}_m^1)$ in $[5m-3,10m-10]$ for every $m \geq 2$. Hence, we have $W' \prec Y'$, from which it follows that $W \prec Y$. Second, we note that by using Equations 4 and 5 it follows that

$$\begin{aligned} y_m \geq 10m-10-|Y'\setminus Y|-(|(\Delta^{-1}(\infty)\cup\Delta^{-1}(\mathbb{Z}_m^2))\cap[5m-3,10m-10]|-t) \geq \\ 10m-10-((k+1-m)+(3m-3-k)-m))-((3m-3-k)-t) = \\ 6m+k+t-5 \geq 3m+(3m-3)+k+t-3 \geq 3m+(k+t)+(k+t)-3. \end{aligned}$$

Hence Equation 6 is satisfied, completing the proof. □

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