On Some Rado Numbers for Generalized Arithmetic Progressions

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Abstract. The 2-color Rado number for the equation $x_1 + x_2 - 2x_3 = c$, which for each constant $c \in \mathbb{Z}$ we denote by $S_1(c)$, is the least integer, if it exists, such that every 2-coloring, $\Delta : [1, S_1(c)] \rightarrow \{0, 1\}$, of the natural numbers admits a monochromatic solution to $x_1 + x_2 - 2x_3 = c$, and otherwise $S_1(c) = \infty$. We determine the 2-color Rado number for the equation $x_1 + x_2 - 2x_3 = c$, when additional inequality restraints on the variables are added. In particular, the case where we require $x_2 < x_3 < x_1$, is a generalization of the 3-term arithmetic progression; and the work done here improves previously established upper bounds to an exact value.

Key words: arithmetic progression, Rado, Ramsey, monochromatic

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1 Introduction

If L is a system of constraints, such as a system of linear equations, then the *r-color Rado number* for L is the least integer N, such that for every r-coloring of $[1,\, N] = \{1, 2, \dots, N\}$, there exists a monochromatic subset, $A \subseteq [1,\, N]$, which satisfies all constraints given by L. One source for the study of Rado numbers is the relatively well-known work of van der Waerden concerning arithmetic progressions. In 1927 Van der Waerden [9], [3] proved the following theorem.

Theorem 1.1. *For integers $m \geq 3$ and $r \geq 2$, there exists a least integer, $N = W(m, r)$, such that every r-coloring of $[1,\, N]$ must contain a monochromatic m-term arithmetic progression.*

In 1933, R. Rado [6], [7], [3] found necessary and sufficient conditions for a system of linear equations to have an r-color Rado number for every $r \geq 2$. Rado's Theorem encompassed Van der Waerden's (see [8] for a short discussion of an observation of Rado which extends his Theorem to certain systems of mixed equalities and inequalities, or [1] for a more specific discussion of how to do this for the case of Van der Waerden's Theorem). For instance, the 3-term arithmetic progressions are the solutions to the following system:

$$\begin{cases} x_2 - x_1 = x_3 - x_2 \\ x_3 - x_2 > 0 \\ x_2 - x_1 > 0 \end{cases}$$

Recently a complete characterization of linear inequality systems that have an r-color Rado number for every $r \geq 2$ was obtained by M. Schäffler [8].

The actual determination of the *van der Waerden numbers*, $W(m, r)$, has proven to be extremely difficult, and is only known for a few small values of m and r [3]. Bialostocki, Lefmann, and Meerdink [2] considered the generalization of the 3-term arithmetic progression obtained by adding a constant to the largest of the three terms—that is they considered the equation $x_1 - x_3 = x_3 - x_2 + c$, $x_2 < x_3 < x_1$. They were able to determine, for $c \geq 10$ even, that the 2-color Rado number for this generalization, $S_5(c)$, satisfies

$$2c + 10 \leq S_5(c) \leq \frac{13}{2}c + 1.$$

More recently, Landman [4] was able to improve their estimate of $S_5(c)$ to

$$2c + 10 \leq S_5(c) \leq \left\lceil \frac{9}{4}c \right\rceil + 9.$$

In this paper, we finish the work of determining the 2-color Rado number for the equation $x_1 + x_2 - 2x_3 = c$, under all possible inequality orderings of the variables (up to symmetry). Consequently, we are able to show that the lower bound first given by Bialostocki, Lefmann, and Meerdink is sharp. As an interesting note, this gives an example of how a Rado-type problem can have an arbitrarily large number of distinct lower bound constructions, which avoid monochromatic solutions, on the interval just one less then the number needed to guarantee a monochromatic solution.

2 The Functions $S_1(c)$, $S_2(c)$, $S_3(c)$ and $S_4(c)$

For every integer c, let $L_1(c)$, $L_2(c)$, $L_3(c)$, $L_4(c)$, $L_5(c)$ represent the systems of equations:

$L_1(c) :\ x_1 + x_2 - 2x_3 = c$

$L_2(c) :\ x_1 + x_2 - 2x_3 = c,\ x_i \neq x_{j,i\neq j}$

$L_3(c) :\ x_1 + x_2 - 2x_3 = c,\ x_1 > x_2 > x_3$

$L_4(c) :\ x_1 + x_2 - 2x_3 = c,\ x_3 > x_2 > x_1$

$L_5(c) :\ x_1 + x_2 - 2x_3 = c,\ x_1 > x_3 > x_2$

For every integer $i \in [1, 5]$, let $S_i(c)$ be the least integer, if it exists, such that every 2-coloring, $\Delta : [1, S_i(c)] \to \{0, 1\}$, of the natural numbers admits a monochromatic solution (x_1, x_2, x_3) to $L_i(c)$. If no such integer exists, let $S_i(c) = \infty$.

We state Theorem 2.1 of D. Schaal and B. Martinelli [5] for completeness. The proof is a much simpler version of the case analysis of Theorem 2.3.

Theorem 2.1. $S_1(c) = \begin{cases} |c|+1 & \text{if } c \text{ is even} \\ \infty & \text{if } c \text{ is odd} \end{cases}$

The next proposition shows that we need only consider c even.

Proposition 2.2. *For $i \in [1, 5]$ and c odd, $S_i(c) = \infty$*

proof. It is easily seen, checking solutions modulo 2, that the coloring, with evens integers colored by 1 and odd integers by 0, has no monochromatic solution.□

Theorem 2.3. *For $c \geq 10$ even, $S_3(c) \leq c + 4$*

proof. Let $\Delta : [1, c + 4] \to \{0, 1\}$ be an arbitrary 2-coloring. Without loss of generality let $\Delta(1) = 0$. Assume by contradiction that Δ avoids a monochromatic solution to $L_3(c)$. We will consider all eight possible colorings of the numbers $\{2,\, c+1,\, c+2\}$, and show in each case that a monochromatic solution follows. The cases are presented in tabular format. The first column lists a solution (x_1, x_2, x_3) to $L_3(c)$; the second column lists the known colorings $(\Delta(x_1),\, \Delta(x_2),\, \Delta(x_3))$ of $\{x_{1,} x_2, x_3\}$, and uses an asterisk when a number's coloring is not yet known; the third column lists the implied coloring of the remaining variable using the assumption that Δ avoids any monochromatic solution. The final entry provides the monochromatic solution for our contradiction.

Case 1: $\Delta(2)=0,\Delta(c+1)=0,\Delta(c+2)=0$		
$(c,2,1)$	$(*,0,0)$	$\Delta(c)=1$
$(c+1,3,2)$	$(0,*,0)$	$\Delta(3)=1$
$(c,6,3)$	$(1,*,1)$	$\Delta(6)=0$
$(c+2,6,4)$	$(0,0,*)$	$\Delta(4)=1$
$(c,8,4)$	$(1,*,1)$	$\Delta(8)=0$
$(c+2,8,5)$	$(0,0,*)$	$\Delta(5)=1$
$(c+3,5,4)$	$(*,1,1)$	$\Delta(c+3)=0$
$(c+3,c+1,\frac{c+4}{2})$	$(0,0,*)$	$\Delta(\frac{c+4}{2})=1$
$(c+4,c,\frac{c+4}{2})$	$(*,1,1)$	$\Delta(c+4)=0$
$(c+4,8,6)$	$(0,0,0)$	

Case 2: $\Delta(2)=0,\Delta(c+1)=0,\Delta(c+2)=1$		
$(c+1,3,2)$	$(0,*,0)$	$\Delta(3)=1$
$(c+2,4,3)$	$(1,*,1)$	$\Delta(4)=0$
$(c+1,7,4)$	$(0,*,0)$	$\Delta(7)=1$
$(c-1,7,3)$	$(*,1,1)$	$\Delta(c-1)=0$
$(c-1,5,2)$	$(0,*,0)$	$\Delta(5)=1$
$(c,2,1)$	$(*,0,0)$	$\Delta(c)=1$
$(c,6,3)$	$(1,*,1)$	$\Delta(6)=0$
$(c-2,6,2)$	$(*,0,0)$	$\Delta(c-2)=1$
$(c+2,c-2,\frac{c}{2})$	$(1,1,*)$	$\Delta(\frac{c}{2})=0$
$(c+1,c-1,\frac{c}{2})$	$(0,0,0)$	

Case 3: $\Delta(2)=0,\Delta(c+1)=1,\Delta(c+2)=0$		
if $\Delta(3)=0$		
$(c,2,1)$	$(*,0,0)$	$\Delta(c)=1$
$(c+2,4,3)$	$(0,*,0)$	$\Delta(4)=1$
$(c-1,3,1)$	$(*,0,0)$	$\Delta(c-1)=1$
$(c+1,c-1,\frac{c}{2})$	$(1,1,*)$	$\Delta(\frac{c}{2})=0$
$(c+2,c-2,\frac{c}{2})$	$(0,*,0)$	$\Delta(c-2)=1$
$(\frac{c+4}{2},\frac{c}{2},1)$	$(*,0,0)$	$\Delta(\frac{c+4}{2})=1$
$(c+4,c,\frac{c+4}{2})$	$(*,1,1)$	$\Delta(c+4)=0$
$(c+4,c+2,\frac{c+6}{2})$	$(0,0,*)$	$\Delta(\frac{c+6}{2})=1$
$(c,c-2,\frac{c-2}{2})$	$(1,1,*)$	$\Delta(\frac{c-2}{2})=0$
$(\frac{c+10}{2},\frac{c-2}{2},2)$	$(*,0,0)$	$\Delta(\frac{c+10}{2})=1$
$(\frac{c+10}{2},\frac{c+6}{2},4)$	$(1,1,1)$	
if $\Delta(3)=1$		
$(c,2,1)$	$(*,0,0)$	$\Delta(c)=1$
$(c,6,3)$	$(1,*,1)$	$\Delta(6)=0$
$(c+1,5,3)$	$(1,*,1)$	$\Delta(5)=0$
$(c+2,8,5)$	$(0,*,0)$	$\Delta(8)=1$
$(c+2,6,4)$	$(0,0,*)$	$\Delta(4)=1$
$(c,8,4)$	$(1,1,1)$	

Case 4: $\Delta(2)=0,\Delta(c+1)=1,\Delta(c+2)=1$		
if $\Delta(4)=1$		
$(c,2,1)$	$(*,0,0)$	$\Delta(c)=1$
$(c+2,c,\frac{c+2}{2})$	$(1,1,*)$	$\Delta(\frac{c+2}{2})=0$
$(\frac{c+6}{2},\frac{c+2}{2},2)$	$(*,0,0)$	$\Delta(\frac{c+6}{2})=1$
$(c+4,c+2,\frac{c+6}{2})$	$(*,1,1)$	$\Delta(c+4)=0$
$(c+2,6,4)$	$(1,*,1)$	$\Delta(6)=0$
$(c,8,4)$	$(1,*,1)$	$\Delta(8)=0$
$(c+4,8,6)$	$(0,0,0)$	
if $\Delta(4)=0$		
$(c-2,4,1)$	$(*,0,0)$	$\Delta(c-2)=1$
$(c+2,c-2,\frac{c}{2})$	$(1,1,*)$	$\Delta(\frac{c}{2})=0$
$(\frac{c+4}{2},\frac{c}{2},1)$	$(*,0,0)$	$\Delta(\frac{c+4}{2})=1$
$(c,2,1)$	$(*,0,0)$	$\Delta(c)=1$
$(c+4,c,\frac{c+4}{2})$	$(*,1,1)$	$\Delta(c+4)=0$
$(c+3,c+1,\frac{c+4}{2})$	$(*,1,1)$	$\Delta(c+3)=0$
$(c+3,5,4)$	$(0,*,0)$	$\Delta(5)=1$
$(c+2,8,5)$	$(1,*,1)$	$\Delta(8)=0$
$(c+4,8,6)$	$(0,0,*)$	$\Delta(6)=1$
$(c+2,10,6)$	$(1,*,1)$	$\Delta(10)=0$
$(c+4,10,7)$	$(0,0,*)$	$\Delta(7)=1$
$(c+2,12,7)(\star)$	$(1,*,1)$	$\Delta(12)=0$
$(c+4,12,8)$	$(0,0,0)$	

($\star$) If $c=10$, then $\Delta(5)=1$ and $\Delta(\frac{c}{2})=0$ is a contradiction, and the last two rows are not needed.

Case 5: $\Delta(2)=1, \Delta(c+1)=0, \Delta(c+2)=0$		
$if\ \Delta(4)=0$		
$(c+2,6,4)$	$(0,*,0)$	$\Delta(6)=1$
$(c-2,6,2)$	$(*,1,1)$	$\Delta(c-2)=0$
$(c-2,4,1)$	$(0,0,0)$	
$if\ \Delta(4)=1$		
$(c,4,2)$	$(*,1,1)$	$\Delta(c)=0$
$(c+2,c,\frac{c+2}{2})$	$(0,0,*)$	$\Delta(\frac{c+2}{2})=1$
$(\frac{c+6}{2},\frac{c+2}{2},2)$	$(*,1,1)$	$\Delta(\frac{c+6}{2})=0$
$(c+4,c+2,\frac{c+6}{2})$	$(*,0,0)$	$\Delta(c+4)=1$
$(c+4,c-2,\frac{c+2}{2})$	$(1,*,1)$	$\Delta(c-2)=0$
$(c+2,c-2,\frac{c}{2})$	$(0,0,*)$	$\Delta(\frac{c}{2})=1$
$(c+4,c-4,\frac{c}{2})$	$(1,*,1)$	$\Delta(c-4)=0$
$(c-4,6,1)(\star\star)$	$(0,*,0)$	$\Delta(6)=1$
$(c+4,8,6)$	$(1,*,1)$	$\Delta(8)=0$
$(c+2,8,5)$	$(0,0,*)$	$\Delta(5)=1$
$(c+4,6,5)$	$(1,1,1)$	

Case 6: $\Delta(2)=1, \Delta(c+1)=0, \Delta(c+2)=1$		
$if\ \Delta(3)=1$		
$(c+2,4,3)$	$(1,*,1)$	$\Delta(4)=0$
$(c-2,4,1)$	$(*,0,0)$	$\Delta(c-2)=1$
$(c-2,6,2)$	$(1,*,1)$	$\Delta(6)=0$
$(c+2,c-2,\frac{c}{2})$	$(1,1,*)$	$\Delta(\frac{c}{2})=0$
$(c+1,c-1,\frac{c}{2})$	$(0,*,0)$	$\Delta(c-1)=1$
$(c-1,7,3)$	$(1,*,1)$	$\Delta(7)=0$
$(c+1,7,4)$	$(0,0,0)$	
$if\ \Delta(3)=0$		
$(c-1,3,1)$	$(*,0,0)$	$\Delta(c-1)=1$
$(c-1,5,2)$	$(1,*,1)$	$\Delta(5)=0$
$(c+1,5,3)$	$(0,0,0)$	

Case 7: $\Delta(2)=1, \Delta(c+1)=1, \Delta(c+2)=0$		
$(c+1,3,2)$	$(1,*,1)$	$\Delta(3)=0$
$(c+2,4,3)$	$(0,*,0)$	$\Delta(4)=1$
$(c,4,2)$	$(*,1,1)$	$\Delta(c)=0$
$(c,6,3)$	$(0,*,0)$	$\Delta(6)=1$
$(c-1,3,1)$	$(*,0,0)$	$\Delta(c-1)=1$
$(c-2,6,2)$	$(*,1,1)$	$\Delta(c-2)=0$
$(c+1,c-1,\frac{c}{2})$	$(1,1,*)$	$\Delta(\frac{c}{2})=0$
$(c+2,c-2,\frac{c}{2})$	$(0,0,0)$	

Case 8: $\Delta(2)=1, \Delta(c+1)=1, \Delta(c+2)=1$		
$if\ \Delta(6)=1$		
$(c+1,3,2)$	$(1,*,1)$	$\Delta(3)=0$
$(c-1,3,1)$	$(*,0,0)$	$\Delta(c-1)=1$
$(c+2,6,4)$	$(1,1,*)$	$\Delta(4)=0$
$(c-2,4,1)$	$(*,0,0)$	$\Delta(c-2)=1$
$(c-2,6,2)$	$(1,1,1)$	
$if\ \Delta(6)=0$		
$(c+1,3,2)$	$(1,*,1)$	$\Delta(3)=0$
$(c-1,3,1)$	$(*,0,0)$	$\Delta(c-1)=1$
$(c,6,3)$	$(*,0,0)$	$\Delta(c)=1$
$(c,4,2)$	$(1,*,1)$	$\Delta(4)=0$
$(c-1,5,2)$	$(1,*,1)$	$\Delta(5)=0$
$(c+3,5,4)$	$(*,0,0)$	$\Delta(c+3)=1$
$(c+1,c-1,\frac{c}{2})$	$(1,1,*)$	$\Delta(\frac{c}{2})=0$
$(c+3,c+1,\frac{c+4}{2})$	$(1,1,*)$	$\Delta(\frac{c+4}{2})=0$
$(\frac{c+4}{2},\frac{c}{2},1)$	$(0,0,0)$	

($\star\star$) If $c=10$, then $\Delta(\frac{c+2}{2})=1$ implies $\Delta(6)=1$ regardless. $\square$

Theorem 2.4. *For $c \geq 10$ even, $S_2(c) \geq c+4$*

proof. Consider the coloring Δ:$[1, c+3] \to \{0,1\}$, where

$$\Delta(t) = \begin{cases} 0 & \text{if } t \geq \frac{c}{2}+3 \text{ or } t=1 \\ 1 & \text{if } 2 \leq t \leq \frac{c}{2}+2 \end{cases}$$

Assume (x_1, x_2, x_3) is a monochromatic solution to $L_2(c)$. If it is of color 1, the maximum of x_1+x_2, which is $(\frac{c}{2}+2)+(\frac{c}{2}+1) = c+3$, is less than the minimum of $c+2x_3$, which is $c+2(2) = c+4$, making it impossible for these to be equal. If it is of color 0 and $x_3 \neq 1$, the maximum of x_1+x_2, which is $(c+3)+(c+2) = 2c+5$, is less than the minimum of $c+2x_3$, which is $c+2(\frac{c}{2}+3) = 2c+6$. If it is of color 0 and $x_3 = 1$, the minimum of x_1+x_2, which is $(\frac{c}{2}+3)+(\frac{c}{2}+4) = c+7$, is greater than $c+2x_3 = c+2(1) = c+2$. So there can be no monochromatic solution under this coloring.□

Lemma 2.1. *For $i \in \{1, 2, 5\}$, $S_i(c) = S_i(-c)$*

proof. Given any constant $c \in \mathbb{Z}$ and arbitrary coloring $\Delta : [1, S_i(-c)] \to \{0, 1\}$, we can induce a coloring, $\Delta' : [1, S_i(-c)] \to \{0, 1\}$, by letting

$$\Delta'(t) = \Delta(S_i(-c) + 1 - t).$$

Since there are $S_i(-c)$ integers, it follows that there exists a monochromatic solution (a, b, d) to the equation $x_1 + x_2 - 2x_3 = -c$, under the Δ' coloring. Thus $a + b - 2d = -c$, and in the Δ coloring, the set

$$\{S_i(-c) + 1 - a,\ S_i(-c) + 1 - b,\ S_i(-c) + 1 - d\}$$

is monochromatic. Since $a + b - 2d = -c$, it follows that

$$(S_i(-c) + 1 - b) + (S_i(-c) + 1 - a) - 2(S_i(-c) + 1 - d) = -(a + b - 2d) = c.$$

Thus the ordered triple $(S_i(-c)+1-b,\ S_i(-c)+1-a,\ S_i(-c)+1-d)$ is a monochromatic solution to $x_1 + x_2 - 2x_3 = c$ under Δ, and we are done if $i = 1$. For $i = 2$ or 5, note that $a > d > b$ [or a and b and d are all distinct] implies that $(S_i(-c) + 1 - b) > (S_i(-c) + 1 - d) > (S_i(-c) + 1 - a)$ [or $(S_i(-c) + 1 - b)$ and $(S_i(-c) + 1 - d)$ and $(S_i(-c) + 1 - a)$ are all distinct], and we see that $S_i(c) \leq S_i(-c)$. Repeating the argument using $-c$ as the constant gives that $S_i(-c) \leq S_i(c)$; and we can conclude that $S_i(c) = S_i(-c)$.□

We next finish the determination of $S_i(c)$ for $i \in [1, 4]$.

Theorem 2.5.
(*i*) *For $c \geq 10$ even, $S_2(c) = S_3(c) = c + 4$*
(*ii*) *For $c \leq -10$ even, $S_2(c) = S_4(c) = -c + 4$*
(*iii*) *For $c \leq 8$, $S_3(c) = \infty$*
(*iv*) *For $c \geq -8$, $S_4(c) = \infty$*

proof. (*i*) It is immediate from their definitions that $S_2(c) \leq S_3(c)$. Hence the result follows from Theorems 2.3 and 2.4.

(ii) Similarly, noting that if (r, s, t) is a solution to $L_3(-c)$, then $(s, r, r+s-t)$ is a solution to $L_4(c)$, it is easily checked that $S_2(c) \leq S_4(c) \leq S_3(-c)$, and so the result follows from (i) and Lemma 2.1.

($iii - iv$) Note that if $x_3 < x_2 < x_1$, then $x_1 + x_2 - 2x_3$ will always be positive. Hence for $c < 0$ there are no solutions to $L_3(c)$. Similarly, if $x_3 > x_2 > x_1$, then $x_1 + x_2 - 2x_3$ will always be negative. Hence for $c > 0$ there are no solutions to $L_4(c)$. Hence from the arguments used in (i) and (ii) and from Proposition 2.2, we note that to cover the remaining cases of the statement it suffices to show $S_3(c) = \infty$ for $c = 0, 2, 4, 6, 8$. If $c \neq 6$, exhaustively checking the solutions to L_3 modulo 4 shows that there are no monochromatic solutions under the coloring

$$\Delta : \mathbb{N} \to \{0,1\},\ \Delta(t) = \begin{cases} 0 & \text{if } t \equiv 1 \,\text{or}\, 2 \bmod (4) \\ 1 & \text{if } t \equiv 0 \,\text{or}\ 3 \bmod (4). \end{cases}$$

If $c = 6$, exhaustively checking the solutions to L_3 modulo 8 shows that there are no monochromatic solutions under the coloring

$$\Delta : \mathbb{N} \to \{0,1\},\ \Delta(t) = \begin{cases} 0 & \text{if } t \equiv 1, 2, 3, \text{or } 4 \mod (8) \\ 1 & \text{if } t \equiv 5, 6, 7, \text{or } 0 \bmod (8). \square \end{cases}$$

3 The Function $S_5(c)$

If $x \equiv y \mod (2)$, then we will denote by $[x, y]_2$ the set $\{z \mid x \leq z \leq y \text{ and } z \equiv x \mod (2)\}$. The main result of the paper is the following.

Theorem 3.1. *For c even, $S_5(c) \leq 2|c| + 10$*

Before we begin the proof of Theorem 3.2, we will need the following two lemmas.

Lemma 3.3. *Let $c \geq 0$ be even. Let $\Delta : [1, N] \to \{0, 1\}$ be a 2-coloring of the integers $[1, N]$ that avoids a monochromatic solution to $L_5(c)$. Let α, $\beta \in [1, N]$, $\alpha \equiv \beta \bmod (2)$, $2 < \alpha < \beta - c - 2$, and $\Delta(\alpha - 2) = \Delta(\beta) \neq \Delta(\alpha)$. Then it follows that*

$$\Delta[\alpha + 1, \frac{\beta - c + \alpha - 2}{2}] = \Delta(\alpha)$$

and

$$\Delta[c + \alpha + 2, \beta - 2] = \Delta(\beta)$$

proof Lemma 3.3. Without loss of generality assume $\Delta(\alpha)$= 1. Since $\alpha < \beta - c - 2$, it follows that the solution $(\beta,\ \alpha-2,\ \frac{\beta-c+\alpha-2}{2})$ implies $\Delta(\frac{\beta-c+\alpha-2}{2}) = 1$; the solution $(\beta-2,\ \alpha,\ \frac{\beta-c+\alpha-2}{2})$ implies $\Delta(\beta-2) = 0$; the solution $(\beta-2,\ \alpha-2,\ \frac{\beta-c+\alpha-4}{2})$ implies $\Delta(\frac{\beta-c+\alpha-4}{2}) = 1$; the solution $(\beta-4,\ \alpha,\ \frac{\beta-c+\alpha-4}{2})$ implies $\Delta(\beta-4) = 0, \ldots,$ and we can continue in this fashion until the solution $(c+\alpha+2,\ \alpha,\ \alpha+1)$ implies $\Delta(c+\alpha+2) = 0$. Thus we see $\Delta[\alpha, \frac{\beta-c+\alpha-2}{2}] = 1$ and $\Delta[c+\alpha+2,\ \beta]_2 = 0$. The set of solutions $\{\ (\beta-3,\ \alpha+1,\ \frac{\beta+\alpha-c-2}{2}),\ (\beta-5,\ \alpha+1,\ \frac{\beta+\alpha-c-4}{2}),\ (\beta-7,\ \alpha+1,\ \frac{\beta+\alpha-c-6}{2}),\ \ldots,\ (c+\alpha+3,\ \alpha+1,\ \alpha+2)\}$, implies that $\Delta[c+\alpha+3,\ \beta-3]_2 = 0$. Thus, since $\Delta[c+\alpha+2,\ \beta]_2 = 0$, it follows that $\Delta[c+\alpha+2,\ \beta-2] = 0$. □

Lemma 3.4. *For $\mu \in \mathbb{Z}$, and $c \geq 0$*

(i) Let $\Delta : [\mu, \mu+(2c+5)] \rightarrow \{0,1\}$ be a 2-coloring that is not a 1-coloring, and let $\alpha \in [\mu, \mu+(2c+5)]$ be the least integer such that $\Delta(\alpha) = 1$.

If $\alpha > \mu+3$, then there is a monochromatic solution (x_1, x_2, x_3) to $L_5(c)$.

(ii) Let $\Delta : [\mu, \mu+(2c+7)] \rightarrow \{0,1\}$ be a 2-coloring that is not a 1-coloring, and let $\alpha \in [\mu, \mu+(2c+7)]$ be the least integer such that $\Delta(\alpha) = 1$.

If $\alpha > \mu+2$, then there is a monochromatic solution (x_1, x_2, x_3) to $L_5(c)$.

proof Lemma 3.4. It is clear that we need only show the lemma is true for $\mu = 1$. (i) Assume to the contrary that there is no monochromatic solution. From the definition of α, it follows that $\Delta[1, \alpha-1] = 0$. Since $\alpha > \mu+2 = 3$, it follows that the solutions $\{(c+3, 1, 2), (c+4, 2, 3), (c+5, 3, 4), \ldots, (c+\alpha, \alpha-2, \alpha-1), (c+\alpha+1, \alpha-3, \alpha-1), (c+\alpha+2, \alpha-4, \alpha-1), \ldots, (c+2\alpha-3, 1, \alpha-1)\}$, all with $x_2, x_3 \in [1, \alpha-1]$, imply $\Delta[c+3, \min\{c+2\alpha-3,\ 2c+6\}] = 1$. Since $\alpha > \mu+2 = 3$, it follows that the set of solutions $\{(2c+5, c+3, c+4),\ (2c+6, c+4, c+5)\}$ implies that $\Delta[2c+5, 2c+6] = 0$.

The solution $(2c+5, 1, \frac{c+6}{2})$ implies that $\Delta(\frac{c+6}{2}) = 1$, and hence $\alpha \leq \frac{c+6}{2}$ and $c+2\alpha-3 \leq 2c+6$. If $c = 0$, then the solution $(2c+5, 1, 3)$ is monochromatic. So we may assume $c > 0$. Since $\alpha \leq \frac{c+6}{2}$, and since $c > 0$, it follows from Lemma 3.3 ($\alpha = \alpha$, $\beta = 2c+5$ or $2c+6$) that $\Delta[\alpha, \frac{\beta-c+\alpha-2}{2}] = \Delta[\alpha, \frac{c+\alpha+3}{2}] = 1$ and $\Delta[c+\alpha+2, 2c+3] = 0$ (if α odd) or $\Delta[\alpha, \frac{\beta-c+\alpha-2}{2}] = \Delta[\alpha, \frac{c+\alpha+4}{2}] = 1$ and $\Delta[c+\alpha+2, 2c+4] = 0$ (if α even). To avoid contradiction, the two intervals $[c+\alpha+2, 2c+3]$ and $[c+3, c+2\alpha-3]$, colored by opposite colors, cannot overlap, which can only occur when $c+2\alpha-3 < c+\alpha+2$, implying $\alpha < 5$. As $\alpha > 4$, this is a contradiction.

(ii) Since the cases $\alpha > 4$ have been handled by part (i), we continue the above proof assuming $\alpha = 4$. In this case, by replacing α with 4 in the intervals from the proceeding paragraphs, we have that: (a) $\Delta[1, \alpha - 1] = \Delta[1, 3] = 0$; (b) $\Delta[\alpha, \frac{c+\alpha+4}{2}] = \Delta[4, \frac{c+8}{2}] = 1$; (c) $\Delta[c + 3, c + 2\alpha - 3] = \Delta[c + 3, c + 5] = 1$; (d) $\Delta([c + \alpha + 2, 2c + 4] \cup [2c + 5, 2c + 6]) = \Delta[c + 6, 2c + 6] = 0$; and (e) $c > 0$.

The solution $(2c+8, c+6, c+7)$ implies $\Delta(2c+8) = 1$. If $c = 2$, then the solution $(2c+8, 4, c+5)$ is monochromatic. So we may assume $c > 2$. The set of solutions $\{(2c+8, c-2, c+3), (2c+8, c, c+4)\}$ implies $\Delta(c - 2) = 0$ and $\Delta(c) = 0$. But then the solution $(2c + 2, c - 2, c)$ is monochromatic, a contradiction. □

proof Theorem 3.2. In light of Lemma 3.1, we need only show that $S_5(c) \leq 2c + 10$ for $c \geq 0$ even. Let $\Delta : [1, 2c + 10] \to \{0, 1\}$ be an arbitrary 2-coloring. Without loss of generality let $\Delta(1) = 0$. Assume by contradiction that Δ avoids a monochromatic solution to $L_5(c)$. We will consider all eight possible colorings of the numbers $\{2, 3, 4\}$, and show in each case that a monochromatic solution follows.

Case 1 and 2: $\Delta(1) = 0$, $\Delta(2) = 0$, $\Delta(3) = 0$, $\Delta(4) = 0$ or 1. By Lemma 3.4 part (ii), since in this case the corresponding $\alpha > 3$, we are assured of a monochromatic solution, contradicting the assumption.

Case 3: $\Delta(1) = 0$, $\Delta(2) = 0$, $\Delta(3) = 1$, $\Delta(4) = 0$. The solution set $\{(c + 3,\ 1,\ 2), (c + 6,\ 2,\ 4), (c + 7,\ 1,\ 4)\}$ implies $\Delta(\{c + 3,\ c + 6,\ c + 7\}) = 1$. The solution set $\{(2c + 9,\ c + 3,\ c + 6), (2c + 8,\ c + 6,\ c + 7)\}$ implies $\Delta[2c + 8, 2c + 9] = 0$. Letting $\alpha = 3$ and $\beta = 2c + 9$, it follows from Lemma 3.3 that $\Delta[3, \frac{2c+9-c+3-2}{2}] = \Delta[3, \frac{c+10}{2}] = 1$, in particular $\Delta(4) = 1$, a contradiction.

Case 4: $\Delta(1) = 0$, $\Delta(2) = 0$, $\Delta(3) = 1$, $\Delta(4) = 1$. Suppose that $\Delta(5) = 0$. Then the solution set $\{(c + 8,\ 2,\ 5), (c + 9,\ 1,\ 5)\}$, implies $\Delta[c + 8, c + 9] = 1$. The solution $(2c + 10,\ c + 8,\ c + 9)$ implies $\Delta(2c + 10) = 0$. By Lemma 3.3 ($\alpha = 4$, $\beta = 2c + 10$), it follows that $\Delta[4, \frac{2c+10-c+4-2}{2}] = \Delta[4, \frac{c+12}{2}] = 1$. In particular, $\Delta(5) = 1$, a contradiction. So $\Delta(5) = 1$. Let $\alpha \in [3, 2c + 10]$ be the least integer such that $\Delta(\alpha) = 0$. It is clear here (and at similar later points in the proof) that such α exists, as if Δ is a 1-coloring, then $(c + 3, 1, 2)$ will be a monochromatic solution. Then letting $\mu = 3$ in Lemma 3.4(ii) gives a monochromatic solution.

Case 5: $\Delta(1) = 0$, $\Delta(2) = 1$, $\Delta(3) = 0$, $\Delta(4) = 0$. The solution set $\{(c+5,\ 1,\ 3), (c+7,\ 1,\ 4)\}$ implies $\Delta(c + 5) = 1$ and $\Delta(c + 7) = 1$. The solution $(2c + 9,\ c + 5,\ c + 7)$ implies $\Delta(2c + 9) = 0$. If $\Delta(5) = 1$, then it follows from Lemma 3.3 ($\alpha = 5$, $\beta = 2c + 9$) that $\Delta[c + \alpha + 2, \beta - 2] = \Delta[c + 7,\ 2c + 7] = 0$, in particular $\Delta(c + 7) = 0$, a contradiction. So $\Delta(5) = 0$. Let $\alpha \in [3, 2c + 10]$ be the least integer such that $\Delta(\alpha) = 1$. Then letting $\mu = 3$ in Lemma 3.4(ii) gives a monochromatic solution.

Case 6: $\Delta(1) = 0$, $\Delta(2) = 1$, $\Delta(3) = 1$, $\Delta(4) = 1$. Let $\alpha \in [2, 2c+9]$ be the least integer such that $\Delta(\alpha) = 0$. Then letting $\mu = 2$ in Lemma 3.4(ii) gives a monochromatic solution.

Case 7: $\Delta(1) = 0$, $\Delta(2) = 1$, $\Delta(3) = 1$, $\Delta(4) = 0$. The solution $(c+7, 1, 4)$ implies $\Delta(c+7) = 1$; $(c+7, 3, 5)$ implies $\Delta(5) = 0$; $(c+6, 4, 5)$ implies $\Delta(c+6) = 1$; $(2c+8, c+6, c+7)$ implies $\Delta(2c+8) = 0$; $(2c+8, 4, \frac{c+12}{2})$ implies $\Delta(\frac{c+12}{2}) = 1$; and $(2c+9, 3, \frac{c+12}{2})$ implies $\Delta(2c+9) = 0$. Letting $\alpha = 3$ and $\beta = 2c+9$, it follows from Lemma 3.3 that $\Delta[c+\alpha+2, \beta-2] = \Delta[c+5, 2c+7] = 0$, in particular $\Delta(c+7) = 0$, a contradiction.

Case 8: $\Delta(1) = 0$, $\Delta(2) = 1$, $\Delta(3) = 0$, $\Delta(4) = 1$.

Subcase a: $\Delta(5) = 1$. The solution set $\{(c+6, 2, 4), (c+8, 2, 5)\}$ implies $\Delta(c+6) = 0$ and $\Delta(c+8) = 0$. The solution $(2c+10,\ c+6,\ c+8)$ implies $\Delta(2c+10) = 1$; $(2c+10, 2,\ \frac{c+12}{2})$ implies $\Delta(\frac{c+12}{2}) = 0$; and $(2c+9,\ 3,\ \frac{c+12}{2})$ implies $\Delta(2c+9) = 1$.

Note that $c \neq 0$, since otherwise either $(c+8, c+6, 7)$ or $(2c+9, 5, 7)$ will be a monochromatic.

Suppose $\Delta(c+9) = 1$. Then the solution $(c+9, 5, 7)$ implies $\Delta(7) = 0$. If $c = 2$, then the solution $(7, 1, 3)$ is monochromatic. So if $\Delta(c+9) = 1$, then we may assume $c > 2$. The solution $(c+11, 3, 7)$ implies $\Delta(c+11) = 1$; $(c+13, 1, 7)$ implies $\Delta(c+13) = 1$; $(c+11, 5, 8)$ implies $\Delta(8) = 0$; $(c+13, 5, 9)$ implies $\Delta(9) = 0$; $(c+10, 8, 9)$ implies $\Delta(c+10) = 1$; $(c+10, 2, 6)$ implies $\Delta(6) = 0$; but then $(c+8, 6, 7)$ is a monochromatic solution. So $\Delta(c+9) = 0$.

The solution $(c+9,\ 3,\ 6)$ implies $\Delta(6) = 1$. If $\Delta(7) = 0$, then the solution $(c+9, 7, 8)$ implies $\Delta(8) = 1$; $(c+11,\ 5,\ 8)$ implies $\Delta(c+11) = 0$; but then the solution $(c+11, 3, 7)$ is monochromatic. So $\Delta(7) = 1$. Let $\alpha \in [4, 2c+9]$ be the least integer such that $\Delta(\alpha) = 0$. Then letting $\mu = 4$ in Lemma 3.4(i) gives a monochromatic solution.

Subcase b: $\Delta(5) = 0$. If $c = 0$, then $(5, 1, 3)$ is monochromatic. So we may assume $c > 0$.

The solution set $\{(c+5, 1, 3),\ (c+6, 2, 4),\ (c+9, 1, 5),\ (c+7, 3, 5)\}$ implies that $\Delta[c+5, c+9]_2 = 1$ and $\Delta(c+6) = 0$. The solution $(2c+9, c+5, c+7)$ implies $\Delta(2c+9) = 0$; $(2c+9,\ 3,\ \frac{c+12}{2})$ implies $\Delta(\frac{c+12}{2}) = 1$; $(2c+10,\ 2,\ \frac{c+12}{2})$ implies $\Delta(2c+10) = 0$; $(2c+10,\ c+6, c+8)$ implies $\Delta(c+8) = 1$; and $(c+8,\ 4,\ 6)$ implies $\Delta(6) = 0$.

Suppose $\Delta(7) = 1$. Letting $\alpha = 7$ and $\beta = 2c+9$, it follows from Lemma 3.3 that $\Delta[c+\alpha+2, \beta-2] = \Delta[c+9, 2c+7] = 0$. In particular, $\Delta(c+9) = 0$, a contradiction. So $\Delta(7) = 0$.

If $c = 2$, the solution $(7, 1, 3)$ is monochromatic. So we may assume $c > 2$.

Suppose $\Delta(8) = 1$. Then the solution $(c + 12, 4, 8)$ implies $\Delta(c + 12) = 0$; $(c + 12, 6, 9)$ implies $\Delta(9) = 1$. The solution $(c+11, 1, 6)$ implies $\Delta(c+11) = 1$. Letting $\alpha = 9$ and $\beta = 2c+9$, it follows from Lemma 3.3 that $\Delta[c + \alpha + 2, \beta - 2] = \Delta[c + 11, 2c + 7] = 0$, in particular $\Delta(c + 11) = 0$, a contradiction. So $\Delta(8) = 0$.

Let $\alpha \in [5, 2c + 10]$ be the least integer such that $\Delta(\alpha) = 1$. Then letting $\mu = 5$ in Lemma 3.4(i) gives a monochromatic solution. □

Theorem 3.5. *For $|c| \geq 10$ even, $S_5(c) = 2|c| + 10$*

proof. Bialostocki, Lefmann, and Meerdink in [2] showed that
$\Delta : [1, 2c + 9] \to \{0, 1\}$, where

$$\Delta(t) = \begin{cases} 0 & \text{if } t \in [1, 2] \cup [6, c + 2] \cup [c + 4, c + 7] \\ 1 & \text{if } t \in [3, 5] \cup \{c + 3\} \cup [c + 8, 2c + 9] \end{cases}$$

avoids a monochromatic solution to $L_5(c)$, for $c \geq 10$ even. Thus $S_5(c) \geq 2c+10$. This matches the upper bound established by Theorem 3.2. Therefore from Lemma 2.1 the desired equality follows for $|c| \geq 10$ even.□

4 Summary and Small c Values

The following table summarizes the results from the previous sections (all odd values are infinite) and includes remaining values for small constants not covered by any of the theorems, which were computed by exhaustive search.

Table 1: Table of 2-Color Rado Numbers for $x_1 + x_2 - 2x_3 = c$, with c even

Ordering	c :	$-c \geq 10$	-8	-6	-4	-2	0	2	4	6	8	$c \geq 10$
none	$S_1(c) =$	$-c + 1$	9	7	5	3	1	3	5	7	9	$c + 1$
$x_i \neq x_j$, $i \neq j$	$S_2(c) =$	$-c + 4$	17	15	13	14	9	14	13	15	17	$c + 4$
$x_1 > x_2 > x_3$	$S_3(c) =$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	$c + 4$
$x_3 > x_2 > x_1$	$S_4(c) =$	$-c + 4$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
$x_1 > x_3 > x_2$	$S_5(c) =$	$-2c + 10$	25	21	17	14	9	14	17	21	25	$2c + 10$

Remarks. It should be noted that the lower bound construction for $S_5(c)$ is not unique. Let $N_5(c)$ denote the number of distinct 2-colorings of $[1, S_5(c) - 1]$ that do not admit a monochromatic solution to $L_5(c)$. In [4] Landman noted that the number of 2-colorings of $[1, 2c+9]$ that do not admit a monochromatic solution to $L_5(c)$ had a tendency to increase with c. This led him to suspect that $S_5(c) > 2c+10$ might hold for sufficiently large c. Theorem 3.5 shows that this is not the case. However, it is easily checked for $c \geq 24$ that the coloring $\Delta : [1, S_5(c)-1] \to \{0,1\}$, given by

$$\Delta(t) = \begin{cases} 0 & \text{if } t \in [1,2] \cup [6, \alpha+5] \cup [\alpha+7, c+2] \cup [c+4, c+7] \\ 1 & \text{if } t \in [3,5] \cup \{\alpha+6\} \cup \{c+3\} \cup [c+8, 2c+9] \end{cases}$$

avoids any monochromatic solution to $L_5(c)$ for $\frac{c+4}{2} \leq \alpha \leq c-10$. Thus we have the interesting fact that

$$\lim_{c\to\infty} N_5(c) = \infty.$$

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