ALGEBRAIC THEORY I HW4
DUE WEDNESDAY NOVEMBER 17, 2021

Question 1. Let R be a ring and let $I = (x_1, \dots, x_n)$ be a non-zero, finitely generated ideal in R, where $x_1, \dots, x_n \in R$. Show that there is an ideal M which is maximal by inclusion among all ideals of R that do not contain I.

For a square-free integer $D \in \mathbb{Z}$, the set $\mathbb{Z}[\sqrt{D}] = \{a + b\sqrt{D} : \ a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$ is a sub-ring of the complex numbers. This is easily checked and you can assume this.

Question 2. Let $D \geq 3$ be a square free integer and let $R = \mathbb{Z}[\sqrt{-D}]$.

(a) Show that 2, $\sqrt{-D}$ and $1+\sqrt{-D}$ are all irreducible elements in R. Hint: Consider the map defined by $N(x) = x\overline{x}$, where $\overline{x} \in \mathbb{C}$ is the complex conjugate of $x \in \mathbb{C}$, and show that $N(xy) = N(x)N(y)$ for x, $y \in R$.

(b) Show that R is not a Unique Factorization Domain (UFD).

Question 3. Let R be a commutative ring and let I, $J \subseteq R$ be ideals. Show that

$$IJ := \{\sum_{i=1}^{n} x_i y_i : \ x_i \in I, y_i \in J, n \geq 0\} \subseteq I \cap J$$

is an ideal of R.

Question 4. Let $R_1, \dots, R_n$ be rings. Then $R = R_1 \times \dots \times R_n$ is ring with multiplication and addition given by

$$(x_1, \dots, x_n)(y_1, \dots, y_n) = (x_1 y_1, \dots, x_n y_n) \quad \text{and} \quad (x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n).$$

If $I \subseteq R$ is an ideal, show that $I = I_1 \times \dots \times I_n$ for some ideals $I_i \subseteq R_i$ for $i = 1, 2, \dots, n$.

Question 5. Let R be a commutative ring. For an ideal $I \subseteq R$, define

$$\sqrt{I} = \{x \in R : \ x^n \in I \text{ for some } n \geq 1\}.$$

(a) Show that $\sqrt{I}$ is an ideal of R containing I.

(b) Show that $\sqrt{I} = R$ if and only if $I = R$.

(c) Show that $\sqrt{M^n} = \sqrt{M} = M$ for any maximal ideal $M \subseteq R$ and $n \geq 1$, where

$$M^n = \underbrace{MM \dots M}_{n} = \{\sum_{i=1}^{m} x_{i,1} \dots x_{i,n} : \ x_{i,j} \in M, m \geq 0\}.$$