ON THREE SETS WITH NONDECREASING DIAMETER

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Abstract. Let $[a, b]$ denote the integers between a and b inclusive and, for a finite subset $X \subseteq \mathbb{Z}$, let $\mathsf{diam}\,(X) = \max(X) - \min(X)$. We write $X <_p Y$ provided $\max(X) < \min(Y)$. For a positive integer m, let $f(m, m, m; 2)$ be the least integer N such that any 2-coloring $\Delta : [1, N] \to \{0, 1\}$ has three monochromatic m-sets B_1, B_2, $B_3 \subseteq [1, N]$ (not necessarily of the same color) with $B_1 <_p B_2 <_p B_3$ and $\mathsf{diam}\,(B_1) \leq \mathsf{diam}\,(B_2) \leq \mathsf{diam}\,(B_3)$. Improving upon upper and lower bounds of Bialostocki, Erdős and Lefmann, we show that $f(m, m, m; 2) = 8m - 5 + \lfloor \frac{2m-2}{3} \rfloor + \delta$ for $m \geq 2$, where $\delta = 1$ if $m \in \{2, 5\}$ and $\delta = 0$ otherwise.

1. Introduction

For a, $b \in \mathbb{R}$, we let $[a, b]$ denote the set of integers between a and b inclusive. For finite subsets X, $Y \subseteq \mathbb{Z}$, the *diameter* of X, denoted by $\mathsf{diam}\,(X)$, is defined as $\max(X) - \min(X)$. Moreover, we say that $X <_p Y$ if and only if $\max(X) < \min(Y)$, meaning all the elements of X come before any element from Y. For positive integers t, m_1, m_2, $\ldots m_t$, r, let $f(m_1, m_2, \ldots, m_t; r)$ be the least integer N such that, for every r-coloring $\Delta : [1, N] \to [0, r-1]$ of the integers $[1, N]$, there exist t subsets B_1, B_2, $\ldots$, $B_t \subseteq [1, N]$ with

(a) each B_i monochromatic, i.e., $|\Delta(B_i)| = 1$ for $i = 1, \ldots, t$,
(b) $|B_i| = m_i$ for $i = 1, 2, \ldots, t$
(c) $B_1 <_p B_2 <_p \ldots <_p B_t$, and
(d) $\mathsf{diam}\,(B_1) \leq \mathsf{diam}\,(B_2) \leq \ldots \leq \mathsf{diam}\,(B_t)$.

A collection of sets B_i that satisfy (a), (b), (c) and (d) is called a *solution* to the problem defined by $p(m_1, m_2, \ldots, m_t; r)$.

The function $f(m_1, m_2, \ldots, m_t; r)$, and the related function $f^*(m_1, m_2, \ldots, m_t; r)$ defined as $f(m_1, m_2, \ldots, m_t; r)$ but requiring the inequalities in (d) to be strict, have been studied by previous authors. Bialostocki, Erdős and Lefmann first introduced $f(m, m, \ldots, m; r)$ in [3], where they determined that $f(m, m; 2) = 5m - 3$, that $f(m, m; 3) = 9m - 7$ and that

$$8m - 4 \leq f(m, m, m; 2) \leq 10m - 6, \tag{1}$$

as well as giving asymptotic bounds for $t = 2$. The problem was motivated in part by zero-sum generalizations in the sense of the Erdős-Ginzburg-Ziv Theorem [6], [9, Theorem 10.1] (see [2], [3], [7] for a short discussion of zero-sum generalizations, including definitions). Subsequently, Bollobás, Erdős, and Jin [4] obtained improved results for $m = 2$, showing that $4r - \log_2 r + 1 \leq f^*(2, 2; r) \leq 4r + 1$ and $f^*(2, 2; 2^k) = 4 \cdot 2^k + 1$, as well as giving improved asymptotic bounds for t and r when $m = 2$. The value of $f(m, m; 4)$ was determined to be $12m - 9$ in [8], the off-diagonal cases (when not all $m_i = m$) are introduced in [14], and other related Ramsey-type problems can also be found in [1], [5], [10], [11], [12], [13].

The goal of this paper is to improve the estimates from (1) to the first exact value for more than two sets. Indeed, we will show that both the upper and lower bounds of Bialostocki, Erdős and Lefmann can be improved, resulting in the value

$$f(m, m, m; 2) = 8m - 5 + \lfloor \frac{2m-2}{3} \rfloor + \delta \quad \text{for } m \geq 2,$$

where $\delta = 1$ if $m \in \{2, 5\}$ and $\delta = 0$ otherwise.

2. Determination of $f(m, m, m; 2)$

Let $\Delta : X \to C$ be a finite coloring of a finite set X by a set of colors C. Let $c \in C$ and $Y \subseteq X$. Let $x_1 < x_2 < \ldots < x_n$ be the integers colored by c in Y. Then, for integers i and j such that $1 \leq i \leq j \leq n$, we use the notation $\mathsf{first}_i^j(c, Y)$ to denote $\{x_i,\, x_{i+1},\, \ldots,\, x_j\}$, which is the set consisting of the i-th through j-th smallest elements of Y colored by c. Likewise, $\mathsf{first}_i(c, Y) = x_i$ is the i-th smallest element colored by c in Y, and $\mathsf{first}(c, Y) = \mathsf{first}_1(c, Y)$ is the first element colored by c in Y. Similarly, we define $\mathsf{last}_i^j(c, Y) = \{x_{n-i+1},\, x_{n-i},\, \ldots,\, x_{n-j+1}\}$ to be the set consisting of the i-th through j-th largest elements of Y colored by c, $\mathsf{last}_i(c, Y) = x_{n-i+1}$ to be the i-th largest element of Y colored by c, and $\mathsf{last}(c, Y) = \mathsf{last}_1(c, Y)$ to be the largest element of Y colored by c. For the sake of simplicity, a coloring $\Delta : [1,\, N] \to C$ will be denoted by the string $\Delta(1)\Delta(2)\Delta(3) \cdots \Delta(N)$, and x^i will be used to denote the string $xx \ldots x$ of length i. Hence $\Delta : [1,\, 6] \to \{0, 1\}$, where $\Delta([1,\, 2]) = \{0\}$, $\Delta(3) = 1$, and $\Delta([4, 6]) = \{0\}$, may be represented by the string $\Delta[1, 6] = 0^2 1 0^3$.

The following technical lemma will help us control the possible 2-colorings of $[1, 3m - 2]$.

Lemma 2.1. *Let $m \geq 2$, let $\Delta : [1, 3m-2] \to \{0, 1\}$ be a 2-coloring and let $B_1 \subseteq [1, 3m-2]$ be a monochromatic m-subset with $\mathsf{diam}\,(B_1) \geq 2m - 2$ satisfying the following additional extremal constraints:*

(a) $\max B_1 := 3m - 2 - \beta$ *is minimal, where* $\beta \in [0, m-1]$*;*
(b) $\mathsf{diam}\,(B_1) := 2m - 2 + \alpha$ *is minimal subject to (a) holding, where* $\alpha \in [0, m - 1 - \beta]$.

Suppose B_1 exists and $\Delta(B_1) = \{1\}$. Then, letting $R = [1, 3m - 2 - \beta]$, one of the following holds.

(i)
- $\beta \leq m - 2$ *and* $|\Delta^{-1}(0) \cap R| \geq m$
- $\Delta R = 1^{m-1-\beta-\nu} H_0 0 H_1 1^{1+\nu}$, *where* $\mu,\, \nu \geq 0$ *are integers*
- $\beta = m - 1 - \alpha$ *or* $\nu = \mu = 0$
- H_1 *is a string of length* $m - 2 - \beta + \mu$ *with exactly* μ *1's and exactly* $m - 2 - \beta$ *0's*
- H_0 *is a string of length* $m - 1 + \beta - \mu$ *containing exactly* $m - 1 - \alpha - \mu$ *1's and exactly* $\alpha + \beta$ *0's*

(ii)
- *either* $\beta < m-1-\alpha$ *or* $\beta = m-1$
- $\Delta R = 0^{m-\alpha-\beta-1}1H_2 1^{m-\beta}$
- H_2 *is a string of length* $m-2+\beta+\alpha$
- *if* $\beta \leq m-2$*, then* $|\Delta^{-1}(0) \cap R| \geq m$
- *if* $\alpha > 0$*, then* $\beta \geq 1$ *and* H_2 *contains exactly* $\beta - 1$ *1's*

(iii)
- $\beta \geq \alpha$
- $|\Delta^{-1}(0) \cap R| < m$
- $\mathsf{first}_m(1, R) \leq 3m-3-\beta-\alpha$
- $|\Delta^{-1}(0) \cap [\mathsf{first}_1(1, R), \mathsf{first}_m(1, R)]| \leq \beta$

Proof. Note (a) and (b) imply

(2) $$\min B_1 = \max B_1 - \mathsf{diam}\, B_1 = 3m-2-\beta-(2m-2+\alpha) = m-\alpha-\beta.$$

Let

$$\eta = 3m-3-\beta-\mathsf{last}_2(1, R) \geq 0$$

be the number of integers colored by 0 between $\mathsf{last}_2(1, R)$ and $\mathsf{last}_1(1, R)$. Let

$$\nu = 3m-3-\beta-\mathsf{last}(0, R) \geq 0$$

be the number of integers strictly between $\mathsf{last}(0, R)$ and $3m-2-\beta$ that are colored by 1. We continue with three claims.

Claim A. If $|\Delta^{-1}(1) \cap R| > m$, then $\mathsf{last}_2(1, R) \leq 3m-3-\beta-\alpha$ and $\eta \geq \alpha$.

If we have $\mathsf{last}_2(1, R) \geq 2m-2+\mathsf{first}(1, R)$, then $B_1' = \mathsf{first}_1^{m-1}(1, R) \cup \{\mathsf{last}_2(1, R)\}$ will be a monochromatic m-subset, in view of the hypothesis $|\Delta^{-1}(1) \cap R| > m$, with $\max B_1' < \max B_1$ and $\mathsf{diam}\, B_1' = \mathsf{last}_2(1, R) - \mathsf{first}(1, R) \geq 2m-2$, contradicting the maximality condition (a) for B_1. Therefore we may instead assume $\mathsf{last}_2(1, R) \leq 2m-3+\mathsf{first}(1, R) \leq 3m-3-\beta-\alpha$, where the final inequality follows from $\mathsf{first}(1, R) \leq \min B_1$ and (2), and now $\eta \geq \alpha$ follows from the definition of η, completing the claim.

Claim B. If $|\Delta^{-1}(0) \cap R| < m$, then either $\beta = m-1$ and (ii) holds or else (iii) holds.

If $\beta = m-1$, then $\alpha = 0$ in view of $\alpha \in [0, m-1-\beta]$. Moreover, $\min B_1 = 1$ by (2), and now (ii) is easily seen to hold. Therefore we need only consider when $\beta \leq m-2$, in which case

(3) $$|\Delta^{-1}(1) \cap R| = 3m-2-\beta-|\Delta^{-1}(0) \cap R| \geq 2m-|\Delta^{-1}(0) \cap R| \geq m+1,$$

where we have utilized the claim hypothesis for the final inequality. We continue by showing that (iii) holds. Claim A implies

(4) $$\mathsf{last}_2(1, R) \leq 3m-3-\beta-\alpha \quad \text{and} \quad \eta \geq \alpha.$$

Moreover, since $\mathsf{first}_m(1, R) \leq \mathsf{last}_2(1, R)$ by (3), we see that (4) also implies

$$\mathsf{first}_m(1, R) \leq 3m-3-\beta-\alpha.$$

By the hypothesis of the claim, we have

$$|\Delta^{-1}(1)\cap[m-\alpha-\beta,3m-2-\beta]|\geq 2m-1+\alpha-|\Delta^{-1}(0)\cap R|\geq m+\alpha. \tag{5}$$

We must have

$$\Delta([1,m-1-\beta-\eta])\subseteq\{0\}, \tag{6}$$

for if $\mathsf{first}(1,R)\leq m-1-\beta-\eta$, then $B_1'=\mathsf{first}_1^{m-1}(1,R)\cup\{\mathsf{last}_2(1,R)\}$ will be a monochromatic m-subset in view of (3) with $\max B_1'<\max B_1$ and $\mathsf{diam}\, B_1'\geq\mathsf{last}_2(1,R)-(m-1-\beta-\eta)=2m-2$ (in view of the definition of η), contradicting the extremal condition (a) for B_1.

From (6), we see there are at least $m-1-\beta-\eta$ integers colored by 0 less than $\mathsf{first}(1,R)$ (with this estimate being rather trivial when $[1,m-1-\beta-\eta]=\emptyset$). By the definition of η, we have at least η integers colored by 0 all greater than $\mathsf{last}_2(1,R)$. In particular, since (3) implies $\mathsf{first}_m(1,R)\leq\mathsf{last}_2(1,R)$, we find that there are at least η integers colored by 0 greater than $\mathsf{first}_m(1,R)$. Since $|\Delta^{-1}(0)\cap R|<m$ holds by the claim hypothesis, this leaves at most $m-1-\eta-(m-1-\beta-\eta)=\beta$ integers that can be colored by 0 between $\mathsf{first}(1,R)$ and $\mathsf{first}_m(1,R)$, i.e.,

$$|\Delta^{-1}(0)\cap[\mathsf{first}_1(1,R),\mathsf{first}_m(1,R)]|\leq\beta.$$

It remains to show

$$\beta\geq\alpha$$

and then (iii) will follow, completing the claim. If $\alpha=0$, then this holds trivially, so we now assume $\alpha\geq 1$, in which case (5) implies there are at least $m+1$ integers colored by 1 in the interval $[m-\alpha-\beta,3m-2-\beta]$. Recall from (2) that $\min B_1=m-\alpha-\beta$. Then we must have

$$\Delta([m-\alpha-\beta+1,m-\beta])=\{0\}, \tag{7}$$

for otherwise

$$B_1'=\mathsf{first}_2^m(1,[m-\alpha-\beta,3m-2-\beta])\cup\{3m-2-\beta\}$$

will be a monochromatic m-subset with $\mathsf{diam}\, B_1>\mathsf{diam}\, B_1'\geq 3m-2-\beta-m+\beta=2m-2$, contradicting the extremal condition (b) for B_1. Hence, we have at least $m-1-\beta-\eta$ integers colored by 0 in $[1,m-1-\beta-\eta]$ by (6), at least α integers colored by 0 in $[m-\alpha-\beta+1,m-\beta]$ by (7), and η integers colored by 0 in $[3m-2-\beta-\eta,3m-3-\beta]$ by the definition of η. Since

$$m-1-\beta-\eta<m-\alpha-\beta+1\leq m-\beta<3m-2-\beta-\eta,$$

where the first inequality follows from (4), the second from $\alpha\geq 1$, and the third from $\eta\leq|\Delta^{-1}(0)\cap R|$ combined with the claim's hypothesis $|\Delta^{-1}(0)\cap R|\leq m-1$, it follows that these three intervals are all disjoint. Thus

$$|\Delta^{-1}(0)\cap R|\geq(m-1-\beta-\eta)+\alpha+\eta=m-1-\beta+\alpha.$$

Combining the above inequality with the claim hypothesis $|\Delta^{-1}(0)\cap R|\leq m-1$ now yields $\beta\geq\alpha$, completing the claim as remarked previously.

Claim C. If $\eta > 0$ and $|\Delta^{-1}(0) \cap R| \geq m$, then

$$\Delta([1, m-1-\beta]) \subseteq \{1\}. \tag{8}$$

Moreover, if we also have $\beta \leq m-2$ and $|\Delta^{-1}(1) \cap R| > m$, then $\eta \geq m-1-\beta$.

From $\eta > 0$ and the definition of η, we have $\Delta(3m-3-\beta) = 0$. From $|\Delta^{-1}(0) \cap R| \geq m$, we conclude that (8) holds, for otherwise $B_1' = \mathsf{first}_1^{m-1}(0, R) \cup \{3m-3-\beta\}$ will be a monochromatic m-subset with $\max B_1' < \max B_1$ and $\mathsf{diam}\, B_1' \geq 3m-3-\beta-(m-1-\beta) = 2m-2$, contradicting the extremal condition (a) for B_1. This completes the first part of the claim. We now assume the hypotheses of the second part. Having proved (8) $\beta \leq m-2$ gives $\Delta(1) = 1$ so that

$$\Delta([2m-1, 3m-3-\beta]) = \{0\};$$

otherwise $B_1' = \mathsf{first}_1^{m-1}(1, R) \cup \{\mathsf{last}_2(1, R)\}$ will be a monochromatic m-subset, since $|\Delta^{-1}(1) \cap R| > m$, with $\max B_1' < \max B_1$ and $\mathsf{diam}\, B_1' \geq (2m-1)-1 = 2m-2$, contradicting the extremal condition (a) for B_1. The claim now follows in view of the definition of η. Having completed the three claims, we continue with the proof of Lemma 2.1.

By Claim B, we may assume

$$|\Delta^{-1}(0) \cap R| \geq m, \tag{9}$$

else the proof is complete. We divide the remainder of the proof into two cases.

Case 1: $\beta = m-1-\alpha$. Note that this is equivalent to $\min B_1 = 1$ by (2).

If $\alpha = 0$, then $\beta = m-1$ and (ii) follows. So assume

$$\alpha \geq 1 \quad \text{and} \quad \beta \leq m-2,$$

where the latter inequality follows from the former in view of the case hypothesis. We will show (i) holds.

Suppose $|\Delta^{-1}(1) \cap R| = m$. Then $|\Delta^{-1}(0) \cap R| = 2m-2-\beta$. In view of (9), we see that, if $|\Delta^{-1}(1) \cap [\mathsf{first}(0, R), \mathsf{last}(0, R)]| > \beta$, then $B_1' = \mathsf{first}_1^{m-1}(0, R) \cup \{\mathsf{last}_1(0, R)\}$ will be a monochromatic m-subset with $\mathsf{diam}\,(B_1') \geq |\Delta^{-1}(0) \cap R| - 1 + \beta + 1 = 2m-2$ and $\max B_1' < \max B_1$, contradicting the maximality condition (a) for B_1. Therefore we must have $|\Delta^{-1}(1) \cap [\mathsf{first}(0, R), \mathsf{last}(0, R)]| \leq \beta$. From the definition of ν, there are precisely $\nu+1$ integers colored by 1 greater than $\mathsf{last}(0, R) = 3m-3-\beta-\nu$ in R. Hence, there are at most $\beta+\nu+1$ elements of R colored by 1 greater than $\mathsf{first}(0, R)$. But now, since $|\Delta^{-1}(1) \cap R| = m$, there must be at least $m-1-\beta-\nu$ integers colored by 1 to the left of $\mathsf{first}(0, R)$. Consequently, if $m-1-\beta-\nu \geq 1$, then (i) follows by letting H_1 be the string given by $\Delta[\mathsf{last}_{m-1-\beta}(0, R)+1, 3m-3-\beta-\nu]$, letting μ be the number of 1's in H_1, and noting that the case hypothesis gives $\beta = m-1-\alpha$ (so that H_0 containing exactly $m-1-\alpha-\mu$ 1's is equivalent to $|\Delta^{-1}(1) \cap R| = m$). On the other hand, if $m-1-\beta-\nu \leq 0$, then let $\nu' = m-2-\beta < \nu$. Since $\beta \leq m-2$, we also have $\nu' \geq 0$, while $m-1-\beta-\nu' = 1$, which is colored by 1 in view of $\min B_1 = 1$. Thus (i) follows using ν' in place of ν by letting H_1 be the string given by $\Delta[\mathsf{last}_{m-1-\beta}(0,R)+1, 3m-3-\beta-\nu']$, and by letting μ be the number of 1's in H_1. So we may now assume

$$|\Delta^{-1}(1)\cap R| > m. \tag{10}$$

In particular, (10) and (9) together force $3m-2-\beta = |R| \geq m+1+m$, implying $\beta \leq m-3$.

In view of (10) and Claim A, it follows that $\eta \geq \alpha > 0$, which, together with (9), allows us to apply Claim C to conclude $\Delta([1, m-1-\beta]) = \{1\}$. Thus, in view of $\beta \leq m-3$, we find that $\Delta(1) = \Delta(2) = 1$, i.e., $\mathsf{first}_2(1,R) = 2$. As a result, $B_1' = \mathsf{first}_2^m(1,R) \cup \{3m-2-\beta\}$ is a monochromatic m-subset with

$$\mathsf{diam}\, B_1' = 3m-4-\beta = 2m-3+\alpha \geq 2m-2,$$

where the second equality follows by the case hypothesis and the inequality from the assumption $\alpha \geq 1$. Since $\max B_1' = \max B_1$ and $2m-2 \leq \mathsf{diam}\, B_1' < \mathsf{diam}\, B_1 = 2m-2+\alpha$, this contradicts the extremal condition (b) for B_1, completing the case.

Case 2: $\beta < m-1-\alpha$. Note this is equivalent to $\min B_1 > 1$ by (2) and implies

$$\beta \leq m-2$$

in view of $\alpha \geq 0$. We divide this case into two subcases.

Subcase 2.1: $\eta = 0$, so that $\mathsf{last}_2(1,R) = 3m-3-\beta$.

We will show that (ii) holds. From the Claim A, it follows that

$$\alpha = 0 \quad \text{ or } \quad |\Delta^{-1}(1)\cap R| = m.$$

From (2), we know there are m integers in R colored by 1, all at least $m-\alpha-\beta$; namely, the m integers from B_1. Thus, if $\mathsf{first}(1,R) < m-\alpha-\beta$, then the set $B_1' = \mathsf{first}_1^{m-1}(1,R) \cup \{\mathsf{last}_2(1,R)\}$ will be a monochromatic m-subset with $\mathsf{diam}\, B_1' \geq 3m-3-\beta-(m-\alpha-\beta-1) = 2m-2+\alpha \geq 2m-2$ and $\max B_1' < \max B_1$, contradicting the extremal condition (a) for B_1. Therefore we may assume $\mathsf{first}(1,R) = \min B_1 = m-\alpha-\beta$, so that

$$\Delta([1, m-\alpha-\beta-1]) = \{0\} \quad \text{ and } \quad \Delta(m-\alpha-\beta) = 1. \tag{11}$$

Now, $[1, m-\alpha-\beta-1]$ is a nonempty interval (in view of the hypothesis of Case 2) entirely colored by 0. Consequently, in view of (9), it follows that the extremal condition (a) for B_1 will be contradicted by $\mathsf{first}_1^{m-1}(0,R) \cup \{\mathsf{last}(0,R)\}$ unless $\Delta([2m-1, 3m-2-\beta]) = \{1\}$. However, in this latter case, (ii) follows (note H_2 containing exactly $\beta-1$ 1's is equivalent to $|\Delta^{-1}(1)\cap R| = m$, and that this in turn forces $\beta - 1 \geq 0$), completing the subcase.

Subcase 2.2: $\eta > 0$

We will show that (i) holds with $\nu = \mu = 0$ in this subcase. By (9), we may apply Claim C to conclude that

$$\Delta([1, m-1-\beta]) = \{1\}. \tag{12}$$

In particular, in view of $\beta \leq m-2$, we see that $\Delta(1) = 1$. Thus, we must have

$$|\Delta^{-1}(1) \cap R| > m, \tag{13}$$

for otherwise $\min B_1 = 1$ by (2), contrary to the hypothesis of Case 2. But now we can further apply Claim C to conclude

$$\eta \geq m-1-\beta \quad \text{and} \quad \Delta([2m-1, 3m-3-\beta]) = \{0\},$$

where the second statement above is simply a restatement of the first in view of the definition of η.

If $\alpha = 0$, then $\min B_1 = m - \beta$ by (2), in which case (12) shows that all integers colored by 0 in R lie between $\min B_1$ and $\max B_1 = 3m-2-\beta$. Thus, in view of (9), it follows that $\mathsf{diam}\, B_1 \geq |B_1| - 1 \geq 2m-1$, contradicting that $\mathsf{diam}\, B_1 = 2m-2+\alpha = 2m-2$ by (b). Therefore we instead conclude that $\alpha \geq 1$.

If $\alpha = 1$, then $\min B_1 = m-\alpha-\beta = m-1-\beta$ by (2). Thus (12) shows that all integers colored by 0 in R lie between $\min B_1$ and $\max B_1 = 3m-2-\beta$. Hence $2m-1 = 2m-2+\alpha = \mathsf{diam}\, B_1 \geq m-1+|\Delta^{-1}(0) \cap R|$, which combined with (9) forces there to be exactly $|\Delta^{-1}(0) \cap R| = m$ integers colored by 0 in $[\min B_1 + 1, \max B_1 - 1] = [m-\beta, 3m-3-\beta]$, and thus exactly $m-2 = m-1-\alpha$ integers colored by 1 in $[m-\beta, 3m-3-\beta]$. As a result, (i) follows with $\mu = \nu = 0$. Therefore we may now assume $\alpha \geq 2$.

Since $\alpha \geq 2$, (12) implies that $\Delta(m-\alpha-\beta+1) = \{1\}$. As a result, recalling from (2) that $\min B_1 = m-\alpha-\beta$, we must have

$$|\Delta^{-1}(1) \cap [m-\alpha-\beta, 3m-2-\beta]| = m,$$

for otherwise $B_1' = \mathsf{first}_2^m(1, [m-\alpha-\beta, 3m-2-\beta]) \cup \{3m-2-\beta\}$ will be a monochromatic m-subset with $\max B_1' = \max B_1$ and $\mathsf{diam}\, B_1' = \mathsf{diam}\, B_1 - 1 = 2m-3+\alpha \geq 2m-2$, contradicting the extremal condition (b) for B_1. But now (i) follows with $\nu = \mu = 0$, completing the proof. □

The next lemma translates the structural information from Lemma 2.1 into the existence of sets with small diameter.

Lemma 2.2. *Let $m \geq 2$ and let $\Delta : [1, 3m-2] \to \{0,1\}$ be a 2-coloring.*

(i) *If there does not exist a monochromatic m-subset B such that $\mathsf{diam}\,(B) \geq 2m-2$, then there exist monochromatic m-subsets D_1, $D_2 \subseteq [1, 3m-2]$ such that*

$$D_1 <_p D_2 \quad \textit{and} \quad \mathsf{diam}\,(D_1) = \mathsf{diam}\,(D_2) = m-1.$$

(ii) *Otherwise, if β, α, ν, μ, and B_1 are as defined in Lemma 2.1, then the following hold.*

(a) *There exist (non necessarily distinct) monochromatic m-subsets A_1, $A_2 \subseteq [1, 3m-2-\alpha-\beta]$ with*

$$\mathsf{diam}\,(A_1) \leq 2m-2-\alpha \qquad \textit{and} \qquad \mathsf{diam}\,(A_2) \leq m + \lfloor \frac{m-1+\beta}{2} \rfloor - 1.$$

(b) *If either Lemma 2.1(iii) holds, or $\alpha \geq 1$ and Lemma 2.1(ii) holds, then*

$$\mathsf{diam}\,A_1 \leq m-1+\beta.$$

(c) *If Lemma 2.1(i) holds, then*

$$\mathsf{diam}\,A_1 \leq 2m-2-\alpha-\mu$$

and there exists a (not necessarily distinct) monochromatic m-subset $A_3 \subseteq [1, m+\alpha+\beta] \subseteq [1, 3m-2-\alpha-\beta]$ with

$$\mathsf{diam}\,A_3 \leq m+\alpha+\beta-1.$$

Proof. First suppose that there does not exist a monochromatic m-subset B with $\mathsf{diam}\,(B) \geq 2m-2$. We may assume each color is used at least m times, for otherwise without loss of generality $D = \mathsf{first}_1^{m-1}(1, [1, 3m-2]) \cup \{\mathsf{last}(1, [1, 3m-2])\}$ is a monochromatic m-set with $\mathsf{diam}\,(D) \geq |D|-1 \geq 2m-2$, contrary to hypothesis. We may without loss of generality assume $\Delta(1) = 1$. Since there does not exist a monochromatic m-subset B with $\mathsf{diam}\,(B) \geq 2m-2$, and since each color is used at least m times, it follows that $\mathsf{last}(1, [1, 3m-2]) \leq 2m-2$, $\Delta(3m-2) = 0$ and $\mathsf{first}(0, [1, 3m-2]) \geq m+1$. Hence $D_1 = [1, m]$ and $D_2 = [2m-1, 3m-2]$ satisfy (i).

So we may now assume there exists a monochromatic m-subset B with $\mathsf{diam}\,(B) \geq 2m-2$. Let β, α, ν, μ, and B_1 be as defined in Lemma 2.1. Let $R = [1, 3m-2-\beta]$ and assume without loss of generality $\Delta(B_1) = \{1\}$. Notice that $\beta \leq m-1$ implies $\beta \leq \lfloor \frac{m-1+\beta}{2} \rfloor$, so that

$$m-1+\beta \leq m + \lfloor \frac{m-1+\beta}{2} \rfloor - 1.$$

Applying Lemma 2.1 to $[1, 3m-2]$ yields three cases.

Case 1: Lemma 2.1(i) holds.

Suppose $\beta = m-1-\alpha$. Then H_0 contains exactly $m-1-\alpha-\mu$ 1's and exactly $\alpha+\beta = m-1$ 0's. Thus the string $H_0 0$ contains m 0's and exactly $m-1-\alpha-\mu = \beta-\mu$ 1's, in which case $A_1 = A_2 = A_3 = \mathsf{first}_1^m(0, R) \subseteq [1, 2m-1-\mu-\nu] \subseteq [1, m+\alpha+\beta] = [1, 3m-2-\alpha-\beta]$ is a monochromatic m-subset with $\mathsf{diam}\,A_1 \leq 2m-2-\alpha-\mu = m-1+\beta-\mu$, as desired. So we may now assume $\beta < m-1-\alpha$, in which case

$$\mu = \nu = 0 \qquad \text{and} \qquad \Delta[1, 3m-2-\alpha-\beta] = 1^{m-1-\beta} H_0 0^{m-\alpha-\beta}. \tag{14}$$

Since H_0 contains exactly $m-1-\alpha$ 1's and exactly $\alpha+\beta$ 0's, it follows from (14) that $A_1 = \mathsf{first}_1^m(0, [1, 3m-2-\alpha-\beta])$ is a monochromatic m-subset such that $\mathsf{diam}\,A_1 \leq m-1+(m-1-\alpha) = 2m-2-\alpha$. Moreover, since $m-1-\beta+m-1-\alpha = 2m-2-\alpha-\beta \geq m$ in view of $\beta < m-\alpha-1$, it follows that there are at least m 1's in the string $1^{m-1-\beta}H_0$. Consequently, since there are at most $\alpha+\beta$ 0's in H_0, it follows that $A_3 = \mathsf{first}_1^m(1,[1,m+\alpha+\beta])$ is a monochromatic m-subset with $\mathsf{diam}\, A_3 \leq m+\alpha+\beta-1$ and $A_3 \subseteq [1,m+\alpha+\beta]$. Since $\frac{1}{2}(\mathsf{diam}\, A_1 + \mathsf{diam}\, A_3) \leq \frac{1}{2}(3m-3+\beta) = m + \frac{m-1+\beta}{2} - 1$, we can take A_2 to be the set from A_1 or A_3 having smaller diameter, and then $\mathsf{diam}\, A_2 \leq m + \lfloor \frac{m-1+\beta}{2} \rfloor - 1$, completing the case.

Case 2: Lemma 2.1(ii) holds.

If $\alpha > 0$, then $\beta \leq m-1-\alpha \leq m-2$, and letting $A_1 = A_2 = \mathsf{first}_1^m(0,R) \subseteq [1,m+\beta] \subseteq [1,3m-2-\alpha-\beta]$, we find that $\mathsf{diam}\, A_1 \leq m-1+\beta \leq 2m-2-\alpha$, as desired. It remains to consider when $\alpha = 0$.

Taking $A_1 = B_1 \subseteq [1,3m-2-\beta] = [1,3m-2-\alpha-\beta]$ gives a monochromatic m-subset with $\mathsf{diam}\, A_1 = \mathsf{diam}\, B_1 = 2m-2+\alpha = 2m-2-\alpha$. If $\beta = m-1$, then $m + \lfloor \frac{m-1+\beta}{2} \rfloor - 1 = 2m-2$, and we may take $A_2 = A_1$. Therefore assume $\beta < m-1$. Assume by way of contradiction that there is no monochromatic m-subset $A_2 \subseteq [1,3m-2-\alpha-\beta] = R$ with $\mathsf{diam}\, A_2 \leq m + \lfloor \frac{m-1+\beta}{2} \rfloor - 1$. Since $\Delta([2m-1,3m-2-\beta]) = \{1\}$, this implies

$$|\Delta^{-1}(1) \cap [2m - \lfloor \frac{m-1+\beta}{2} \rfloor - 1 - \beta, 2m-2]| \leq \beta - 1, \tag{15}$$

else $A_2 = \mathsf{last}_1^m(1,R)$ will be such a set. On the other hand, since $\Delta([1,m-1-\beta]) = \{0\}$ and $|\Delta^{-1}(0) \cap R| \geq m$, it likewise follows that

$$|\Delta^{-1}(0) \cap [m-\beta, m + \lfloor \frac{m-1+\beta}{2} \rfloor]| \leq \beta, \tag{16}$$

else $A_2 = \mathsf{first}_1^m(0,R)$ will be such a set. Observe that

$$2m - \lfloor \frac{m-1+\beta}{2} \rfloor - 1 - \beta \geq 2m - \lfloor \frac{m-1+m-2}{2} \rfloor - 1 - \beta = m - \beta + 1 \quad \text{and}$$
$$m + \lfloor \frac{m-1+\beta}{2} \rfloor \leq m + \lfloor \frac{m-1+m-2}{2} \rfloor = 2m-2,$$

both in view of $\beta \leq m-2$. Thus, letting $R' = [2m - \lfloor \frac{m-1+\beta}{2} \rfloor - 1 - \beta, m + \lfloor \frac{m-1+\beta}{2} \rfloor]$, we see that (15) and (16) yield the contradiction

$$2\beta \leq 2\lfloor \frac{m-1+\beta}{2} \rfloor - m + 2 + \beta = |R'| \leq 2\beta - 1,$$

completing the case.

Case 3: Lemma 2.1(iii) holds.

Letting $\mathsf{first}_1^m(1,[1,3m-2-\beta-\alpha]) = A_1 = A_2$, it follows that $\mathsf{diam}\,(A_1) \leq m-1+\beta \leq 2m-2-\alpha$, completing the proof. □

We are now ready to present our main result.

Theorem 2.1. *Let $m \geq 2$ be an integer. Then*

$$f(m,m,m;2) = 8m - 5 + \lfloor \frac{2m-2}{3} \rfloor + \delta,$$

where $\delta = 1$ *if* $m \in \{2, 5\}$ *and* $\delta = 0$ *otherwise.*

Proof. Observe that $f(m, m, m; 2) \geq 8m - 5 + \lfloor \frac{2m-2}{3} \rfloor$ follows for $m \geq 2$ by considering the coloring of $[1, 8m - 6 + \lfloor \frac{2m-2}{3} \rfloor]$ given by the string

$$01^{m-1}0^{m-1}1^{m-1}0^{\lfloor \frac{2m-2}{3} \rfloor}1^{m-\lfloor \frac{2m-2}{3} \rfloor -1}0^{m-1}1^{2m-1+\lfloor \frac{2m-2}{3} \rfloor}0^{m-1}.$$

Likewise, $f(2, 2, 2; 2) \geq 12$ follows by considering the coloring of $[1, 11]$ given by the string

$$10101101110,$$

and $f(5, 5, 5; 2) \geq 38$ follows by considering the coloring of $[1, 37]$ given by the string

$$01^4 0^4 1^4 0^8 1^4 0^2 1^7 0^3.$$

We proceed to show that $f(m, m, m; 2) \leq 8m - 5 + \lfloor \frac{2m-2}{3} \rfloor + \delta$, where $\delta = 1$ for $m \in \{2, 5\}$ and $\delta = 0$ otherwise.

Suppose by way of contradiction that $\Delta[-2m+2, 6m-4+\lfloor \frac{2m-2}{3} \rfloor + \delta] \to \{0, 1\}$ is a 2-coloring that avoids a monochromatic solution to $p(m, m, m; 2)$ (the problem is translation invariant, so we choose our interval to begin with $-2m + 2$ for notational convenience in the proof). From the pigeonhole principle, it follows that there is a monochromatic m-set

$$A_0 \subseteq [-2m + 2, 0] \quad \text{with} \quad \mathsf{diam}\,(A_0) \leq 2m - 2.$$

The strategy is to show that any 2-coloring of $[1, 6m - 4 + \lfloor \frac{2m-2}{3} \rfloor + \delta]$ must either contain a monochromatic solution to $p(m, m, m; 2)$ or else a monochromatic solution $B <_p C$ to $p(m, m; 2)$ with $\mathsf{diam}\, B \geq 2m - 2$. In the latter case, A_0, B and C will give the desired monochromatic solution to $p(m, m, m; 2)$.

Apply Lemma 2.2 to $[1, 3m - 2]$. If (i) holds, then, since the pigeonhole principle guarantees there is a monochromatic m-set $D_3 \subseteq [3m - 1, 5m - 3]$, we find that D_1, D_2 and D_3 form a monochromatic solution to $p(m, m, m; 2)$. So we may assume (ii) of Lemma 2.2 holds and we will apply it and Lemma 2.1 to $[1, 3m - 2]$. Let α, β, ν, μ, A_1, A_2, A_3 and B_1 be as in Lemmas 2.2 and 2.1. Let $\Delta(B_1) = \{c_1\}$ with $\{c_1, c_0\} = \{1, 0\}$. Thus, when reading the conclusion of Lemma 2.1, we must use c_1 in place of 1 and c_0 in place of 0. Recall that we have the trivial inequality

$$0 \leq \alpha + \beta \leq m - 1$$

in view of the definition of α. Let

$$R_1 = [1, 3m - 2 - \beta] \quad \text{and} \quad R_2 = [3m - 1 - \beta, 6m - 4 + \lfloor \frac{2m-2}{3} \rfloor + \delta].$$

We need to find three sets with nondecreasing diameter. We would like our first set to have as small diameter as possible and also be pushed up into the very beginning of our interval as much as possible. At the moment, the best we can do is use the pigeonhole principle to obtain A_0 above. We want our second set to have larger diameter than A_0 but by as little as possible and also be pushed as far forward in the interval R_1 as possible. At the moment, B_1 is our best candidate and the parameters β and α reflect how far forward it is pushed and how small its diameter is. This makes R_2 the realm for finding a third set with larger diameter than B_1. The sets A_1, A_2 and A_3 from Lemma 2.2 are also available for use. They have smaller diameters than A_0 but at the cost of being much farther forward in the interval. Of course, R_2 is is not long enough to finish the proof by these means, but we can gain some important information about how the integers are colored there. There are two basic ways the coloring of R_2 can thwart us in the search for a third set of larger diameter. Each color class with at least m elements must have all its integers tightly grouped together, which will help us later. However, we first need to rule out the other way to avoid a third set of large diameter, namely, when one color class has less than m integers in R_2. We do that in the first step below. As we will see, this coloring strategy, while not the most difficult case to deal with in the proof, includes an optimal strategy for $m = 2$. If R_2 had one more term, the case becomes even easier, but when the parameters all achieve extremal values, a separate argument is needed to finish the case, only valid for large enough m, which takes into account the structure of ΔR_1 and ΔR_2.

Step 1: $|\Delta^{-1}(0) \cap R_2| \geq m$ and $|\Delta^{-1}(1) \cap R_2| \geq m$.

Suppose without loss of generality that $|\Delta^{-1}(0) \cap R_2| < m$. Then

$$|\Delta^{-1}(1) \cap R_2| \geq 2m - 1 + \beta + \lfloor \frac{2m-2}{3} \rfloor + \delta.$$

Let

$$\gamma = |\Delta^{-1}(0) \cap [\mathsf{first}(1, R_2), \mathsf{last}(1, R_2)]|.$$

If $\alpha \leq \lfloor \frac{2m-2}{3} \rfloor + \gamma + \beta + \delta$, then A_0, B_1, and $C = \mathsf{first}_1^{m-1}(1, R_2) \cup \mathsf{last}(1, R_2)$ form a monochromatic solution to $p(m, m, m; 2)$. So we may assume

$$\alpha \geq \lfloor \frac{2m-2}{3} \rfloor + \gamma + \beta + \delta + 1 \geq \frac{2m-1}{3} + \delta + \gamma + \beta. \tag{17}$$

Let y be the least integer such that $\Delta(y) = 1$, $y \geq \mathsf{first}(1, R_2) + 2m - 2 - \alpha$ and $y \geq \mathsf{first}_m(1, R_2)$, and let

$$C_1 = \mathsf{first}_1^{m-1}(1, R_2) \cup \{y\}.$$

Note that y exists in view of $|\Delta^{-1}(1) \cap R_2| \geq 2m - 1 - \alpha \geq m$. Observe that

$$|\Delta^{-1}(1) \cap [\mathsf{first}(1, R_2), y]| \leq \max\{2m - 1 - \alpha,\ m\} = 2m - 1 - \alpha, \tag{18}$$

since otherwise the minimality of y is contradicted by $y' = \mathsf{first}_{2m-1-\alpha}(1, R_2)$. Then

$$2m - 2 - \alpha \leq \mathsf{diam}\, C_1 \leq 2m - 2 - \alpha + \gamma \leq \frac{4m-5}{3} - \delta - \beta, \tag{19}$$

where the latter inequality follows from (17). By (18), there are at least

$$2m - 1 + \beta + \lfloor \frac{2m-2}{3} \rfloor + \delta - (2m - 1 - \alpha) = \lfloor \frac{2m-2}{3} \rfloor + \alpha + \beta + \delta$$

integers colored by 1 in

$$R_3 = [y + 1, 6m - 4 + \lfloor \frac{2m-2}{3} \rfloor + \delta].$$

In view of (17), we have $\lfloor\frac{2m-2}{3}\rfloor+\alpha+\beta+\delta\geq m$. Thus, letting $C_2=\mathsf{first}_1^{m-1}(1,R_3)\cup\{\mathsf{last}(1,R_3)\}$, it follows from (17) that

$$\mathsf{diam}\, C_2\geq|\Delta^{-1}(1)\cap R_3|-1\geq\frac{2m-4}{3}+\alpha+\beta+\delta-1\geq\frac{4m-5}{3}+2\delta+2\beta+\gamma-1\geq\frac{4m-5}{3}-1.$$

Comparing the above bound with that of (19), we see that if any estimate used in obtaining these bounds can be improved by 1, then A_1, C_1 and C_2 will be a monochromatic solution to $p(m,m,m;2)$. Assuming this is not the case, we instead find that $\delta=0$, $\beta=0$, $\gamma=0$, and $|\Delta^{-1}(1)\cap R_2|=2m-1+\beta+\lfloor\frac{2m-2}{3}\rfloor+\delta$.

Since $\delta=0$, we have

$$m\geq 3.$$

Since $\delta=\gamma=\beta=0$ and $|\Delta^{-1}(1)\cap R_2|=2m-1+\beta+\lfloor\frac{2m-2}{3}\rfloor+\delta$, it follows that there are exactly $2m-1+\lfloor\frac{2m-2}{3}\rfloor$ integers colored by 1 in R_2, all of them consecutive, with the remaining $m-1$ integers colored by 0. Thus

$$\Delta R_2=0^{\lambda}1^{2m-1+\lfloor\frac{2m-2}{3}\rfloor}0^{m-1-\lambda},\tag{20}$$

for some $\lambda\in[0,m-1]$. Since $\alpha=\lfloor\frac{2m+1}{3}\rfloor>0=\beta$, we cannot have (iii) holding in Lemma 2.1. Since $\alpha>0$ and $\beta=0$, we cannot have (ii) holding in Lemma 2.1. Thus Lemma 2.1(i) must hold.

If $\beta=m-1-\alpha$, then $\alpha=m-1$ in view of $\beta=0$. Moreover, the string $H_0 0 H_1$ contains exactly $m-1-\alpha=\beta=0$ c_1's. Thus $\Delta R_1=c_1^{m-1-\nu}c_0^{2m-2}c_1^{1+\nu}$. But now, in view of $m\geq 3$ and (20), it is clear that $[m-\nu,2m-\nu-1]$, $[\mathsf{first}(1,R_2),\mathsf{first}(1,R_2)+m-1]$ and $[\mathsf{first}(1,R_2)+m,\mathsf{first}(1,R_2)+2m-1]$ are three monochromatic sets each of diameter $m-1$, yielding a monochromatic solution to $p(m,m,m;2)$. Therefore we may instead assume $0=\beta<m-1-\alpha$, implying

$$\mu=\nu=\beta=0,\quad \alpha=\lfloor\frac{2m+1}{3}\rfloor\leq m-2\quad\text{and}\quad m\geq 5,\tag{21}$$

where the final inequality follows from the second. In this case, Lemma 2.1 implies

$$\Delta R_1={c_1}^{m-1}H_0c_0^{m-1}c_1.\tag{22}$$

We divide the remainder of the proof of Step 1 into two cases.

Case 1.1: $c_0=1$.

In this case, we see from (20) and (22) that

$$\Delta[2m-1,6m-4-\lfloor\frac{2m-2}{3}\rfloor]=1^{m-1}0^{\lambda+1}1^{2m-2+\alpha}0^{m-1-\lambda}.\tag{23}$$

Recall from (21) that

$$3\leq\alpha=\lfloor\frac{2m+1}{3}\rfloor\leq m-2$$

and recall from Lemma 2.2(ii)(c) that $A_3\subseteq[1,m+\alpha+\beta]\subseteq[1,2m-2]$, where the second inclusion follows from (21), with $\mathsf{diam}\, A_3\leq m+\alpha+\beta-1=m+\alpha-1$.

Observe that $m + \lambda + 2m - 2 + \alpha \geq 3m - 2 + \alpha \geq 2m + 2\alpha$ in view of $\alpha \leq m - 2$. Consequently, if $\lambda + 1 \leq \alpha$, then it follows from (23) that $C = [2m-1, 3m-3] \cup \{3m - 2 + \alpha\}$ and $D = [3m - 1 + \alpha, 4m - 3 + \alpha] \cup \{4m - 2 + 2\alpha\}$ will be monochromatic m-subsets with $\mathsf{diam}\, C = \mathsf{diam}\, D = m + \alpha - 1$, in which case A_3, C and D form a monochromatic solution to $p(m, m, m; 2)$. Therefore we may instead assume

$$\lambda + 1 \geq \alpha + 1. \tag{24}$$

Observe that $2m-3+\alpha \geq 2m \geq m+\lambda+1$ in view of $\alpha \geq 3$ and $\lambda \leq m-1$. Consequently, it follows from (23) that $C = [2m-1, 3m-3] \cup \{3m+\lambda-1\}$ and $D = [3m+\lambda, 4m+\lambda-2] \cup \{4m+2\lambda\}$ are monochromatic m-subsets with $\mathsf{diam}\, C = \mathsf{diam}\, D = m + \lambda \geq m + \alpha$ (in view of (24)), in which case A_3, C and D form a monochromatic solution to $p(m, m, m; 2)$, completing the case.

Case 1.2: $c_0 = 0$

In this case, we see from (20) and (22) that

$$\Delta[2m-1, 6m-4-\lfloor \frac{2m-2}{3} \rfloor] = 0^{m-1}10^{\lambda}1^{2m-2+\alpha}0^{m-1-\lambda}. \tag{25}$$

Recall from Lemma 2.2(ii)(c) that $A_3 \subseteq [1, m+\alpha+\beta] \subseteq [1, 2m-2]$ is a monochromatic m-subset, where the second inclusion follows from (21), with $\mathsf{diam}\, A_3 \leq m + \alpha - 1$ (in view of $\beta = 0$).

If $\lambda \geq \alpha$, then recall that $\alpha \geq 1$ and observe that $2m-2+\alpha \geq m+\alpha$. As a result, we see from (25) that $C = [2m-1, 3m-3] \cup \{3m-2+\alpha\}$ and $D = [3m-1+\lambda, 4m-3+\lambda] \cup \{4m-2+\lambda+\alpha\}$ are monochromatic m-subsets with $\mathsf{diam}\, C = \mathsf{diam}\, D = m + \alpha - 1$, in which case A_3, C, and D form a monochromatic solution to $p(m, m, m; 2)$.

If $m - \alpha - 1 \leq \lambda \leq \alpha - 1$, then $m - 1 + \alpha \geq m + \lambda$. As a result, we see from (25) that $C = \{3m - 2\} \cup [3m - 1 + \lambda, 4m - 3 + \lambda]$ and $D = [4m - 2 + \lambda, 5m - 4 + \lambda] \cup \{5m - 3 + 2\lambda\}$ are monochromatic m-subsets with $\mathsf{diam}\, C = \mathsf{diam}\, D = m + \lambda - 1 \geq 2m - 2 - \alpha$, in which case $A_1 \subseteq [1, 3m - 2 - \alpha] \subseteq [1, 3m - 3]$, C and D form a monochromatic solution to $p(m, m, m; 2)$.

Finally, if $\lambda \leq m-\alpha-1$, then it follows from (25) that $C = \{3m-2\} \cup [4m-2-\alpha, 5m-4-\alpha]$ is a monochromatic m-subset with $\mathsf{diam}\, C = 2m - 2 - \alpha$. Moreover, there are

$$2m - 2 + \alpha - (2m - 1 - \alpha - 1 - \lambda) = 2\alpha + \lambda \geq 2\alpha \geq 2m - 1 - \alpha$$

integers greater than $5m - 4 - \alpha$ that are colored by 1, where the final inequality follows in view of $3\alpha = 3\lfloor \frac{2m+1}{3} \rfloor \geq 3\frac{2m-1}{3} = 2m - 1$. Thus, in view of (25), we conclude that $D = [5m-3-\alpha, 6m-5-\alpha] \cup \{\mathsf{last}(1, R_2)\}$ is a monochromatic m-subset with $\mathsf{diam}\, D \geq 2m-2-\alpha$, in which case $A_1 \subseteq [1, 3m - 2 - \alpha] \subseteq [1, 3m - 3]$, C and D form a monochromatic solution to $p(m, m, m; 2)$, completing the case and Step 1.

With Step 1 complete, we can now focus on the more difficult coloring strategy. Nonetheless, knowing that each color class has at least m elements in R_2 forces some nice consequences and allows us to define the pair of sets C_1 and C_2 below as well as get our first estimate for how far the color class 0 extends in R_2. For certain values of our parameters, the sets C_1 and C_2 allow us to complete the proof immediately, which results in important restrictions for α and β, cases that must be dealt in the remainder of the proof. In particular, Lemma 2.1(iii) will be ruled out entirely, which explains why we spent so little effort parameterizing the coloring of R_1 in this case. Pulling back the first integer colored by 0 will come into play later in the proof, and the initial estimate established below will be needed to prime later arguments which, in certain cases, will be used to improve this estimate even further.

We may without loss of generality assume $\Delta(3m-1-\beta)=1$. Then, since $|\Delta^{-1}(1)\cap R_1|\geq m$ by Step 1, we must have $\Delta([5m-3-\beta+\alpha, 6m-4+\lfloor\frac{2m-2}{3}\rfloor+\delta])=\{0\}$ else, letting $C=\mathsf{first}_1^{m-1}(1,R_2)\cup\{\mathsf{last}(1,R_2)\}$, it follows that A_0, B_1 and C form a monochromatic solution to $p(m,m,m;2)$. But now we likewise have $\Delta([3m-1-\beta, 4m-2-\alpha+\lfloor\frac{2m-2}{3}\rfloor+\delta])=\{1\}$ else, letting $C=\mathsf{first}_1^{m-1}(0,R_2)\cup\{\mathsf{last}(0,R_2)\}$, it follows that A_0, B_1 and C form a monochromatic solution to $p(m,m,m;2)$. In summary,

$$\Delta\Big([3m-1-\beta, 4m-2-\alpha+\lfloor\frac{2m-2}{3}\rfloor+\delta]\Big)=\{1\} \quad \text{and} \tag{26}$$

$$\Delta\Big([5m-3-\beta+\alpha, 6m-4+\lfloor\frac{2m-2}{3}\rfloor+\delta]\Big)=\{0\}. \tag{27}$$

Note that both of these intervals contain $m-\alpha+\beta+\lfloor\frac{2m-2}{3}\rfloor+\delta$ integers.

Let

$$\begin{aligned} C_1 &= [3m-1-\beta, 4m-3-\beta]\cup\{4m-2-\alpha+\lfloor\frac{2m-2}{3}\rfloor+\delta\} \quad \text{and} \\ C_2 &= [5m-3-\beta+\alpha, 6m-5-\beta+\alpha]\cup\{6m-4+\lfloor\frac{2m-2}{3}\rfloor+\delta\}. \end{aligned}$$

When $\beta-\alpha+\lfloor\frac{2m-2}{3}\rfloor+\delta\geq 0$, these are both monochromatic m-subsets of diameter $\mathsf{diam}\,C_1=\mathsf{diam}\,C_2=m-1+\beta-\alpha+\lfloor\frac{2m-2}{3}\rfloor+\delta$.

If $\beta-\alpha+\lfloor\frac{2m-2}{3}\rfloor+\delta\geq m-1-\alpha$, then C_1 and C_2 are monochromatic m-subsets with $\mathsf{diam}\,C_1=\mathsf{diam}\,C_2\geq 2m-2-\alpha$, in which case A_1, C_1 and C_2 form a monochromatic solution to $p(m,m,m;2)$. If $\beta-\alpha+\lfloor\frac{2m-2}{3}\rfloor+\delta\geq\frac{m-2+\beta}{2}$, then C_1 and C_2 are monochromatic m-subsets with $\mathsf{diam}\,C_1=\mathsf{diam}\,C_2\geq m+\lceil\frac{m-2+\beta}{2}\rceil-1$, in which case A_2, C_1 and C_2 form a monochromatic solution to $p(m,m,m;2)$. In summary, we may instead assume the contrary of both these inequalities or we are done. Hence, we now consider

$$\beta\leq\frac{m-2}{3}-\delta<m-1 \quad \text{and} \tag{28}$$

$$\alpha\geq\frac{m+1}{6}+\frac{\beta}{2}+\delta>0. \tag{29}$$

From (29) and (28), we derive that

$$\alpha>\frac{m-2}{6}+\frac{\beta}{2}=\frac{1}{2}(\frac{m-2}{3}+\beta)\geq\beta. \tag{30}$$

Thus Lemma 2.1(iii) cannot hold.

If $\mathsf{first}(0, R_2) > 5m-2-\beta$, then $\Delta([3m-1-\beta, 5m-2-\beta]) = \{1\}$, in which case $D_1 = [3m-1-\beta, 4m-2-\beta]$, $D_2 = [4m-1-\beta, 5m-2-\beta]$ and $D_3 = \mathsf{first}_1^m(0, R_2)$ form a monochromatic solution to $p(m, m, m; 2)$ in view of Step 1. Therefore we may instead assume

$$\mathsf{first}(0, R_2) \leq 5m-2-\beta. \tag{31}$$

From the above work, we know either (i) or (ii) holds in Lemma 2.1. Lemma 2.1(i) contains the two stubborn cases we will spend the rest of proof dealing with: the cases $\beta = m-1-\alpha$ and $\nu = \mu = 0$. The case $\beta = m-1-\alpha$ reflects a kind of equilibrium between the parameters α and β, a coloring of R_1 without any "slack," where the diameter of B_1 and that of R_1 coincide. The case $\mu = \nu = 0$ is more akin to that dealt with in Step 1. This case corresponds to when the coloring of R_1 uses a minimal number of integers colored by 1 in R_1, a strategy that allows the set B_1 to not necessarily fit tightly into the interval R_2. The next step rules out Lemma 2.1(ii), allowing us to focus our attentions on the difficult equilibrium cases of Lemma 2.1(i) for the remainder of the proof, and is divided into two subcases depending on whether $c_1 = 1$ or 0.

Step 2: Lemma 2.1(i) holds.

Since Lemma 2.1(iii) does not hold as noted above, assume to the contrary that Lemma 2.1(ii) holds instead. We divide the step into two cases.

Case 2.1: $c_1 = 1$.

In this case, Lemma 2.1(ii) and (26) yield

$$\Delta\Big([2m-1, 4m-2-\alpha+\lfloor\frac{2m-2}{3}\rfloor+\delta]\Big) = \{1\}.$$

Recall that $\alpha+\beta \leq m-1$. Hence, letting $C = [3m-1-\alpha-\beta, 4m-3-\alpha-\beta] \cup \{4m-2-\alpha\}$, it follows that C is a monochromatic m-set with $\mathsf{diam}\,(C) = m-1+\beta$. Consequently, we must have $\mathsf{first}(0, R_2) \geq 5m-2-\beta+\lfloor\frac{2m-2}{3}\rfloor+\delta$, for otherwise, letting $D = \mathsf{first}_1^{m-1}(0, R_2) \cup \{6m-4+\lfloor\frac{2m-2}{3}\rfloor+\delta\}$, it follows in view of Step 1 and Lemma 2.2(ii)(b) that $A_1 \subseteq [1, 3m-2-\alpha-\beta]$, C and D form a monochromatic solution to $p(m, m, m; 2)$. However, since $\lfloor\frac{2m-2}{3}\rfloor+\delta \geq 1$, this is contrary to (31), completing the case.

Case 2.2: $c_1 = 0$.

In this case, we have $\beta \geq 1$ (in view of (29) and Lemma 2.1(ii)) and

$$\Delta R_1 = 1^{m-\alpha-\beta-1}0H_2 0^{m-\beta}$$

with $\Delta[m-\alpha-\beta+1, 2m-2] = H_2$ a string of length $m-2+\beta+\alpha$ that contains exactly $\beta-1$ 0's and $m-1+\alpha$ 1's (in view of $\alpha > 0$ from (29)). In particular, $\mathsf{last}_{m-1}(1, R_1) \geq m-\beta+1$.

Let $y = \max\{i \in [1, m-\beta+1] : \Delta(i) = 1\}$. Since there are at most $\beta-1$ integers colored by 0 in $[m-\alpha-\beta+1, m-\beta+1] \subseteq [m-\alpha-\beta+1, 2m-2]$ (with the inclusion in view of $\beta \geq 1$), which is an interval of length $\alpha+1 \geq \beta+2$ (in view of (30)), it follows that

$$m-2\beta+2 \leq y \leq m-\beta+1. \tag{32}$$

Let $B=\{y\}\cup \mathsf{last}_1^{m-2}(1,R_1)\cup\{3m-1-\beta\}$. Since $\mathsf{last}_{m-1}(1,R_1)\geq m-\beta+1$, it follows in view of (32) that B is a monochromatic m-subset with

$$2m-2\leq \mathsf{diam}\,(B)\leq 2m-3+\beta.$$

If $\mathsf{first}(0,R_2)\leq 4m-1+\lfloor\frac{2m-2}{3}\rfloor-\beta+\delta$, then A_0, B and $\mathsf{first}_1^{m-1}(0,R_2)\cup\{\mathsf{last}(0,R_2)\}$ form a monochromatic solution to $p(m,m,m;2)$ in view of Step 1 and (27). Therefore we may instead assume

$$\mathsf{first}(0,R_2)\geq 4m+\lfloor\frac{2m-2}{3}\rfloor-\beta+\delta>4m-2,$$

with the latter inequality in view of (28). Hence, letting $C=[3m-1-\beta,4m-3-\beta]\cup\{4m-2\}$, it follows that C is a monochromatic m-subset with $\mathsf{diam}\,C=m-1+\beta$. On the other hand, in view of Step 1, (31) and (27), we have $D=\mathsf{first}_1^{m-1}(0,R_2)\cup\{\mathsf{last}(0,R_2)\}$ being a monochromatic m-subset with

$$\mathsf{diam}\,D\geq m-2+\beta+\lfloor\frac{2m-2}{3}\rfloor+\delta\geq m+\beta-1.$$

Thus A_1, C and D form a monochromatic solution to $p(m,m,m;2)$ in view of Lemma 2.2(ii)(b) and (29), completing Step 2.

Step 2 now allows us to focus on the main and most difficult case of the proof: Lemma 2.1(i). The pattern that will eventually emerge is that the parameters ν and μ generally only help us, albeit while at the same time forcing us to keep track of them since it is not immediately evident that this will be the case. The amount we will lose from arguments early on in the proof when they are non-zero is more than made up by how they augment later arguments. Below, we take a pause to improve the estimates done before Step 2 utilizing the fact that we now know Lemma 2.1(i) holds. These improvements might seem minor at first, but given how razor thin our margin is for dealing with the case of Lemma 2.1(i), they are actually quite vital.

Recall, in view of (26) and (27), that C_1 and C_2 are both monochromatic m-subsets of diameter $\mathsf{diam}\,C_1=\mathsf{diam}\,C_2=m-1+\beta-\alpha+\lfloor\frac{2m-2}{3}\rfloor+\delta$ when $\beta-\alpha+\lfloor\frac{2m-2}{3}\rfloor+\delta\geq 0$.

If $\beta-\alpha+\lfloor\frac{2m-2}{3}\rfloor+\delta\geq m-1-\alpha-\mu$, then C_1 and C_2 are monochromatic m-subsets with $\mathsf{diam}\,C_1=\mathsf{diam}\,C_2\geq 2m-2-\alpha-\mu$, in which case A_1, C_1 and C_2 form a monochromatic solution to $p(m,m,m;2)$ in view of Lemma 2.2(ii)(c) and Step 2. Likewise, if $\beta-\alpha+\lfloor\frac{2m-2}{3}\rfloor+\delta\geq\alpha+\beta$, then C_1 and C_2 are monochromatic m-subsets with $\mathsf{diam}\,C_1=\mathsf{diam}\,C_2\geq m+\alpha+\beta-1$, in which case A_3, C_1 and C_2 form a monochromatic solution to $p(m,m,m;2)$. In summary, we may instead assume the contrary of both these inequalities, in turn yielding

$$\beta\leq\frac{m-2}{3}-\mu-\delta\quad\text{ and }\tag{33}$$

$$\alpha\geq\frac{m+\delta}{3}>0.\tag{34}$$

In particular, (33) and $\beta\geq 0$ yield

$$m\geq 3.$$

Next, we rule out the case when $c_1 = 0$, handling two subcases depending on which of the two possibilities of Lemma2.1(i) holds. Hidden in this case is the optimal coloring for $m = 5$, which explains the lengthy extension of Case 3.2 even after most estimates of parameters are forced to hold with equality. We begin to see more subtle arguments, such as the definition of y, that reflect that need to break the intervals into harder to define halves than R_1 and R_2 or the mild variations seem so far. The trickiest arguments that remain in the proof make use of integers like y (or z and z' later) defined precisely so that the distribution of integers colored by 1 and 0 splits well to either side of them.

Step 3: $c_1 = 1$.

Assume by way of contradiction that $c_1 = 0$. We divide the step into two cases.

Case 3.1: $\beta = m - 1 - \alpha$.

In this case, Lemma 2.1(i) yields

$$\Delta[1, 3m - 2 - \beta] = 0^{m-1-\beta-\nu} H_0 1 H_1 0^{1+\nu}$$

with the string $\Delta[m - \beta - \nu, 2m - 2 - \nu - \mu] = H_0$ containing exactly $\beta - \mu = m - 1 - \alpha - \mu$ 0's. We trivially have $m - 1 - \beta - \nu \geq 0$ and $m - 1 - \alpha - \mu = \beta - \mu \geq 0$ as these quantities from Lemma 2.1(i) must be nonzero, implying $\nu + \mu \leq m - 1$. Thus $2m - 2 - \nu - \mu \geq m - 1$, which means the string $H_0 1$ covers the interval $[m - \beta - \nu, m]$, thus ensuring there are at most $\beta - \mu$ integers colored by 0 in $[m - \beta - \nu, m]$. Since this interval contains at least $\beta + 1$ elements, this ensures that there is some

$$y \in [m - \beta + \mu, m] \quad \text{with} \quad \Delta(y) = 1 \text{ and } y \leq \mathsf{last}_{m-1-\beta}(1, R_1), \tag{35}$$

where the inequality follows by recalling from Lemma 2.1(i) that the string H_1 contains exactly $m - 2 - \beta$ 1's. From (26) and $\alpha = m - 1 - \beta$, we know

$$\Delta([3m - 1 - \beta, 3m - 1 + \beta + \lfloor \frac{2m - 2}{3} \rfloor + \delta]) = \{1\}.$$

Thus $B = \{y\} \cup \mathsf{last}_1^{m-2-\beta}(1, R_1) \cup [3m - 1 - \beta, 3m - 1]$ is a monochromatic m-subset (in view of (35)) with

$$2m - 1 \leq \mathsf{diam}\, B \leq 2m - 1 + \beta.$$

If $\mathsf{first}(0, R_2) \leq 4m - 3 - \beta + \lfloor \frac{2m-2}{3} \rfloor + \delta$, then A_0, B and $\mathsf{first}_1^{m-1} \cup \{\mathsf{last}(0, R_2)\}$ form a monochromatic solution to $p(m, m, m; 2)$ in view of Step 1 and (27). Therefore we may instead assume

$$\mathsf{first}(0, R_2) \geq 4m - 2 - \beta + \lfloor \frac{2m - 2}{3} \rfloor + \delta > 4m - 2,$$

with the latter inequality following from (33). Thus $C = [3m - 1 - \beta, 4m - 3 - \beta] \cup \{4m - 2\}$ is a monochromatic m-subset with $\mathsf{diam}\, C = m - 1 + \beta$. On the other hand, in view of Step 1, (27) and (31), we see that $D = \mathsf{first}_1^{m-1}(0, R_2) \cup \{\mathsf{last}(0, R_2)\}$ is a monochromatic m-subset with

$\mathsf{diam}\, D \geq m-2+\beta+\lfloor\frac{2m-2}{3}\rfloor+\delta \geq m-1+\beta$, whence A_1, C and D give a monochromatic solution to $p(m,m,m;2)$ in view of Lemma 2.2(ii) and $\alpha = m-1-\beta$, completing the case.

Case 3.2: $\beta < m-1-\alpha$.

In this case,

$$\alpha \leq m-2-\beta \leq m-2, \tag{36}$$

while Lemma 2.1(i) yields $\mu=\nu=0$ and

$$\Delta[1,3m-2-\beta] = 0^{m-1-\beta}H_0 1^{m-1-\beta}0 \tag{37}$$

with the string $\Delta[m-\beta, 2m-2]=H_0$ containing exactly $m-1-\alpha$ 0's and exactly $\alpha+\beta$ 1's. Now $m-1-\beta \geq \alpha+1$ by the case hypothesis, and $4m-2-\alpha+\lfloor\frac{2m-2}{3}\rfloor \geq 5m-3-2\alpha$ in view of (34). Thus, if $\alpha \geq 2$, then (37), (36) and (26) imply $B=[3m-1-\alpha-\beta, 3m-3-\beta]\cup[4m-3-\alpha-\beta, 5m-3-2\alpha-\beta]$ is a monochromatic m-subset with $\mathsf{diam}\, B = 2m-2-\alpha$ and $B\subseteq[3m-1-\alpha-\beta, \mathsf{first}(0,R_2)-1]$. On the other hand, if $\alpha\leq 1$, then (34) ensures that $\alpha=1$, $m=3$ (recall that we now know $m\geq 3$), $\beta=0$ (in view of the case hypothesis $\beta\leq m-2-\alpha$), and $\mathsf{diam}\, A_3 = m+\alpha+\beta-1 = 2m-2-\alpha$ (by Lemma 2.2(ii)). In this case, $B=\{3m-3-\beta\}\cup[4m-3-\alpha-\beta, 5m-5-\alpha-\beta]$ is a monochromatic m-subset with $\mathsf{diam}\, B = 2m-2-\alpha$ and $\min B \geq m+\alpha+\beta+1$.

In view of Step 1, (27) and (31), it follows that $C=\mathsf{first}_1^{m-1}(0,R_2)\cup\{\mathsf{last}(0,R_2)\}$ is a monochromatic m-subset with

$$\mathsf{diam}\, C \geq m-2+\beta+\lfloor\frac{2m-2}{3}\rfloor+\delta \geq m-2+\lfloor\frac{2m-2}{3}\rfloor \geq 2m-3-\alpha, \tag{38}$$

with the latter inequality once more in view of (34). Thus A_1 (if $\alpha\geq 2$) or A_3 (if $\alpha=1$), B and C will form a monochromatic solution to $p(m,m,m;2)$ unless equality holds in all the estimates used to derive (38). In particular, we must have $\delta=0$, $\beta=0$, $\alpha=\lfloor\frac{m+1}{3}\rfloor$ and $\mathsf{first}(0,R_2)=5m-2-\beta=5m-2$.

Since $\beta=0$, (37) gives

$$\Delta R_1 = 0^{m-1}H_0 1^{m-1}0.$$

In view of $\mathsf{first}(0,R_2)=5m-2$ and $\beta=0$, we have

$$\Delta([3m-1,5m-3])=\{1\}.$$

Thus, since $\alpha\leq\frac{m+1}{3}$ and $m\geq 3$ imply $4m-2+2\alpha\leq 4m-2+\frac{2m+2}{3}<5m-2$, it follows that $D_1=[2m-1,3m-3]\cup\{3m-2+\alpha\}$ and $D_2=[3m-1+\alpha, 4m-3+\alpha]\cup\{4m-2+2\alpha\}$ are monochromatic m-subsets with $\mathsf{diam}\, D_1=\mathsf{diam}\, D_2 = m-1+\alpha$. So $A_3\subseteq[1,m+\alpha+\beta]\subseteq[1,2m-2]$ (the inclusion follows from the case hypothesis), D_1 and D_2 form a monochromatic solution to $p(m,m,m;2)$ in view of Lemma 2.2(ii)(c) and $\beta=0$, completing the case and Step 3.

In view of Steps 2 and 3 and Lemma 2.1(i), we now have

$$\Delta[1, 3m-2-\beta] = 1^{m-1-\beta-\nu} H_0 0 H_1 1^{1+\nu}, \tag{39}$$

where $H_0 = \Delta[m-\beta-\nu, 2m-2-\mu-\nu]$ is a string containing exactly $m-1-\alpha-\mu$ 1's and $\alpha+\beta$ 0's and $H_1 = \Delta[2m-\mu-\nu, 3m-3-\beta-\nu]$ is a string containing exactly μ 1's and $m-2-\beta$ 0's. Let

$$R_2' = [3m-2-\beta-\nu, 6m-4+\lfloor\frac{2m-2}{3}\rfloor]$$

and observe that $\Delta([3m-2-\beta-\nu, 4m-2-\alpha+\lfloor\frac{2m-2}{3}\rfloor+\delta]) = \{1\}$ in view of (26) and (39).

If $\mathsf{first}(0, R_2) > 5m-3-\beta-\nu$, then $\Delta([3m-2-\beta-\nu, 5m-3-\beta-\nu]) = \{1\}$, in which case $D_1 = [3m-2-\beta-\nu, 4m-3-\beta-\nu]$, $D_2 = [4m-2-\beta-\nu, 5m-3-\beta-\nu]$ and $D_3 = \mathsf{first}_1^m(0, R_2)$ form a monochromatic solution to $p(m,m,m;2)$ in view of Step 1. Therefore we may instead assume

$$\mathsf{first}(0, R_2) \le 5m-3-\beta-\nu. \tag{40}$$

Let

$$\gamma_0 = |\Delta^{-1}(0) \cap [\mathsf{first}(1, R_2), \mathsf{last}(1, R_2)]|.$$

We must have

$$\gamma_0 \le |\Delta^{-1}(0)\cap R_2| + \alpha - m - \lfloor\frac{2m-2}{3}\rfloor - \beta \le |\Delta^{-1}(0)\cap R_2| + \alpha - \frac{5m-4}{3} - \beta, \tag{41}$$

for otherwise $C = \mathsf{first}_1^{m-1}(1, R_2) \cup \{\mathsf{last}(1, R_2)\}$ will be a monochromatic m-subset (in view of Step 1) with

$$\begin{aligned} \mathsf{diam}\,(C) &= |\Delta^{-1}(1)\cap R_2| + \gamma_0 - 1 = (3m-2+\beta+\lfloor\frac{2m-2}{3}\rfloor + \delta - |\Delta^{-1}(0)\cap R_2|) + \gamma_0 - 1 \\ &\ge 2m-2+\alpha, \end{aligned}$$

in which case A_0, B_1 and C form a monochromatic solution to $p(m,m,m;2)$.

With Step 3 complete, we are now stuck in the trickiest remaining case. For later arguments to work, it is essential that we first improve the estimates for α and ν, which we do below in Step 4.

Step 4: $\nu \le \alpha - 1$ and $\alpha \ge \frac{2m+2}{3} + \mu + \nu + \delta$.

To simplify notation, we proceed in two cases.

Case 4.1: $\beta = m - \alpha + 1$.

In this case, Lemma 2.2(ii)(c) shows $\mathsf{diam}\, A_1 \leq m + \beta - \mu - 1$ with $A_1 \subseteq [1, 3m - 2 - \alpha - \beta]$. Suppose $\nu \geq \alpha - 1 - \lfloor \frac{2m-2}{3} \rfloor - \mu - \delta$. Then (39) and (26) imply that the interval

$$I = [3m - 1 - \alpha - \beta + \mu + \lfloor \frac{2m-2}{3} \rfloor + \delta, 4m - 2 - \alpha + \lfloor \frac{2m-2}{3} \rfloor + \delta]$$

is entirely colored by 1. Since $|I| \geq m+\beta-\mu$, it follows that $B = \mathsf{first}_1^{m-1}(1, I) \cup \{\mathsf{first}_{m+\beta-\mu}(1, I)\}$ is a monochromatic m-subset with $\mathsf{diam}\, B = m + \beta - \mu - 1$ and $\min B \geq 3m - 1 - \alpha - \beta$. Thus, in view of (40), (27) and Step 1, we see that A_1, B and $\mathsf{first}^{m-1}(0, R_2) \cup \{\mathsf{last}(0, R_2)\}$ form a monochromatic solution to $p(m, m, m; 2)$. So we may instead assume

$$\nu \leq \alpha - 2 - \lfloor \frac{2m-2}{3} \rfloor - \mu - \delta \leq \alpha - 2,$$

which rearranges to yield the desired bound for α.

Case 4.2: $\beta < m - \alpha + 1$.

In this case, Lemma 2.1(i) implies that $\mu = \nu = 0$, so that $\nu \leq \alpha - 1$ follows by (34). In particular, $\min R_2' \geq 3m - 1 - \alpha - \beta$. Assume by way of contradiction that

$$\alpha \leq \frac{2m+1}{3} + \delta. \tag{42}$$

Then $4m - 3 \leq 4m - 2 - \alpha + \lfloor \frac{2m-2}{3} \rfloor + \delta$, so that (39) and (26) ensure

$$\Delta([3m - 2 - \beta, 4m - 3]) = \{1\}, \tag{43}$$

which is an interval of length $m + \beta \geq m$. Let $y = \mathsf{last}_2(1, R_1)$. Then $y \leq 2m - 2$ by (39). If $y < 2m - 2 - \beta$, then $D_1 = [2m - 2 - \beta, 3m - 3 - \beta]$ is a monochromatic m-subset (by (39)) with $\mathsf{diam}\, D_1 = m - 1$, in which case D_1, $\mathsf{first}_1^m(1, R_2')$ and $\mathsf{first}_1^m(0, R_2')$ form a monochromatic solution to $p(m, m, m; 2)$ in view of Step 1 and (43). Therefore, we instead have $2m - 2 - \beta \leq y \leq 2m - 2$, which means $D_2 = \{y\} \cup [3m - 1, 4m - 3]$ is a monochromatic m-subset (by (43)) with

$$2m - 1 \leq \mathsf{diam}\, D_2 \leq 2m - 1 + \beta.$$

Thus, in view of Step 1 and (27), we must have

$$\mathsf{first}(0, R_2) \geq 4m - 2 - \beta + \lfloor \frac{2m-2}{3} \rfloor + \delta,$$

else A_0, D_2 and $\mathsf{first}_1^{m-1}(0, R_2) \cup \{\mathsf{last}(0, R_2)\}$ form a monochromatic solution to $p(m, m, m; 2)$. Consequently, since $5m - 4 - \beta - \alpha \leq 4m - 3 - \beta + \lfloor \frac{2m-2}{3} \rfloor + \delta$ (by (34)), we see that $D_3 = [3m-2-\beta, 4m-4-\beta] \cup \{5m-4-\beta-\alpha\}$ is a monochromatic m-subset with $\mathsf{diam}\, D_3 = 2m-2-\alpha$. Moreover, in view of Step 1, (40) and (27), we see that $D_4 = \mathsf{first}_1^{m-1}(0, R_2) \cup \{\mathsf{last}(0, R_2)\}$ is a monochromatic m-subset with

$$\mathsf{diam}\, D_4 \geq m - 1 + \beta + \frac{2m-4}{3} + \delta \geq m - 1 + \frac{2m-4}{3} > 2m - 3 - \alpha,$$

where the latter inequality follows in view of (34). Thus A_1, D_3 and D_4 form a monochromatic solution to $p(m,m,m;2)$ (in view of Lemma 2.2(ii) and $\alpha \geq 1$), completing the case and Step 4.

Step 4 allows us to, once more, improve some previous estimates in rather vital ways summarized below. This marks the second time we have improved the estimates for α and β.

In view of Step 4 and the basic inequality $\alpha + \beta \leq m-1$, we have

$$m \geq 6 \quad \text{and} \quad \delta = 0,$$

$R_2' \subseteq [3m-1-\alpha-\beta, 6m-4+\lfloor\frac{2m-2}{3}\rfloor]$,

$$\alpha \geq \frac{2m+2}{3} + \mu + \nu \quad \text{and} \quad \beta \leq \frac{m-5}{3} - \mu - \nu. \tag{44}$$

With the improved estimates above, we are now ready to begin the intricate argument finishing the last remaining case. We first have to rule out the case when the end of the interval R_2 is colored by a long string of 0's.

Step 5: $\mathsf{last}(1, R_2) \geq 5m-2-\alpha-\beta$.

Assume to the contrary that $\mathsf{last}(1, R_2) \leq 5m-3-\alpha-\beta$. Then

$$I = [5m-2-\alpha-\beta, 6m-4+\lfloor\frac{2m-2}{3}\rfloor]$$

is an interval entirely colored by 0 with

$$|I| = m-1+\alpha+\beta+\lfloor\frac{2m-2}{3}\rfloor. \tag{45}$$

In particular, $|I| \geq m-1+\frac{2m+2}{3}+\beta+\frac{2m-4}{3} \geq 2m+\beta+\frac{m-5}{3} \geq 2m+2\beta$ by (44). Consequently, if $\beta = m-1-\alpha$, then A_1, $\mathsf{first}_1^{m-1}(0,I) \cup \{\mathsf{first}_{m+\beta}(0,I)\}$ and $\mathsf{first}_{m+\beta+1}^{2m+\beta-1}(0,I) \cup \{\mathsf{first}_{2m+2\beta}(0,I)\}$ will form a monochromatic solution to $p(m,m,m;2)$ in view of Lemma 2.2(ii)(c). Therefore we may instead assume $\beta < m-1-\alpha$, in which case Lemma 2.1(i) implies $\mu = \nu = 0$.

In view of $\beta < m-1-\alpha$, we have $\min I \geq 4m$. Thus we must have

$$|\Delta^{-1}(1) \cap [3m-2-\beta, 4m-3]| \leq m-1, \tag{46}$$

for otherwise $\mathsf{first}_1^m(1, R_2')$, $\mathsf{first}_1^{m-1}(0,I) \cup \{\mathsf{first}_{m+\beta}(0,I)\}$ and $\mathsf{first}_{m+\beta+1}^{2m+\beta-1}(0,I) \cup \{\mathsf{first}_{2m+2\beta}(0,I)\}$ will form a monochromatic solution to $p(m,m,m;2)$. But this means

$$\mathsf{first}_{\beta+1}(0, R_2) \leq 4m-3. \tag{47}$$

By (39) and $\mu = 0$, we know $\Delta[2m-1, 3m-3-\beta] = 0^{m-1-\beta}$. Thus

$$C = \mathsf{first}_1^{m-1}(0, [2m-1, 6m-4+\lfloor\frac{2m-2}{3}\rfloor]) \cup \{5m-2-\alpha-\beta\}$$

is a monochromatic m-subset (in view of (47)) with $\mathsf{diam}\, C = 3m-3-\alpha-\beta \geq 2m-1$ (in view of $\beta < m-1-\alpha$). On the other hand,

$$D = [5m-1-\alpha-\beta, 6m-3-\alpha-\beta] \cup \{6m-4+\lfloor\frac{2m-2}{3}\rfloor\}$$

is a monochromatic m-subset (in view of (45)) with $\mathsf{diam}\, D = m-3+\alpha+\beta+\lfloor\frac{2m-2}{3}\rfloor$. However, in view of $\beta \geq 0$ and (44), we have $m-3+\alpha+\beta+\lfloor\frac{2m-2}{3}\rfloor \geq 3m-3-\alpha-\beta$. Hence A_0, C and D form a monochromatic solution to $p(m,m,m;2)$, completing Step 5.

Step 5 allows us to define the integer y below, which will play a vital role in the remainder of the proof, including the definition of the interval R_3. We also introduce the quantity β' and the various γ_0 parameters describing the distribution of 0's in R_3, in turn allowing us to describe a new important set, C_0, whose diameter is precisely controlled by the relevant parameters.

Let $\beta' = m-1-\alpha \geq \beta$. In view of (44), we have

$$\beta \leq \beta' \leq \frac{m-5}{3} - \mu - \nu. \tag{48}$$

Let y be the least integer such that $\Delta(y) = 1$, $y \geq \mathsf{first}_m(1, R_2')$ and

$$y \geq \mathsf{first}(1, R_2') + m + \beta' - \mu - 1 = 5m - 4 - \alpha - \beta - \mu - \nu.$$

Note y exists in view of Steps 1 and 5. Let

$$R_3 = [3m - 2 - \beta - \nu, y]$$

and let

$$\gamma_0' = |\Delta^{-1}(0) \cap R_3| \quad \text{and} \quad \gamma_0'' = |\Delta^{-1}(0) \cap [\mathsf{first}(1, R_3), \mathsf{last}_2(1, R_3)]|.$$

Note that $\mathsf{first}(1, R_3) = \min R_3 = \min R_2' = 3m - 2 - \beta - \nu$.

If $|\Delta^{-1}(1) \cap R_3| > m$, then we must have $|\Delta^{-1}(1) \cap R_3| - 1 + \gamma_0'' \leq m + \beta' - \mu - 1$, else the minimality of y will be contradicted by $\mathsf{last}_2(1, R_3)$. Thus

$$|\Delta^{-1}(1) \cap R_3| \leq m + \max\{0, \beta' - \mu - \gamma_0''\} \leq m + \beta' - \mu. \tag{49}$$

Consequently, if $\mathsf{last}(0, R_3) - \mathsf{first}(0, R_3) \geq m - 2 + \max\{\gamma_0', \beta' - \mu\}$, then $[\mathsf{first}(0, R_3), \mathsf{last}(0, R_3)]$ is an interval of length at least $m - 1 + \gamma_0'$ that, by definition of γ_0', can contain at most γ_0' integers colored by 0 and, consequently, must contain at least $m-1$ integers colored by 1. Since $y > \mathsf{last}(0, R_3)$ and is also colored by 1, this would mean there are no more than $|\Delta^{-1}(1) \cap R_3| - m \leq \beta' - \mu \leq m - 1 - \alpha$ (with the second inequality by (49)) integers colored by 1 in $[\min R_3, \mathsf{first}(0, R_3) - 1] = [3m - 2 - \beta - \nu, \mathsf{first}(0, R_3) - 1]$. However, by (26) and (39), we know the first $m - \alpha + \beta + \lfloor\frac{2m-2}{3}\rfloor + 1 + \nu \geq m - \alpha$ consecutive integers of R_3 are colored by 1, making this impossible. Therefore we instead conclude that $\mathsf{last}(0, R_3) - \mathsf{first}(0, R_3) \leq m - 3 + \max\{\gamma_0', \beta' - \mu\}$, in which case it is easily seen that there is a monochromatic m-subset $C_0 \subseteq [3m - 2 - \beta - \nu, y] \subseteq [3m - 1 - \alpha - \beta, y]$ (where the second inclusion follows from Step 4) with

$$\mathsf{diam}\, C_0 = m - 1 + \max\{\gamma_0', \beta' - \mu\} \tag{50}$$

(since $\mathsf{diam}\, R_3 \geq m - 1 + \max\{\gamma_0', \beta' - \mu\}$ in view of the definitions of R_3, y and γ_0').

Next, akin to Step 1, we need to ensure enough integers are colored by 0 in R_2 for later arguments. In particular, we want to know there are at least m integers colored by 0 all greater than z, making it possible to form a monochromatic m-set colored by 0 with all terms coming after z.

Step 6: $|\Delta^{-1}(0) \cap R_2| \geq m + \beta + 1$.

Suppose by way of contradiction that $|\Delta^{-1}(0) \cap R_2| \leq m + \beta$, so that

$$|\Delta^{-1}(1) \cap R_2'| \geq (3m - 1 + \beta + \frac{2m-4}{3} + \nu) - m - \beta = \frac{8m-7}{3} + \nu. \tag{51}$$

Then it follows from (49) and (51) that there are at least $\frac{8m-7}{3} - m - \beta' + \mu + \nu = \frac{5m-7}{3} - \beta' + \mu + \nu$ integers colored by 1 that are greater than y. Thus we must have

$$\frac{5m-7}{3} - \beta' + \mu + \nu \leq m - 1 + \max\{\gamma_0', \beta' - \mu\}, \tag{52}$$

for otherwise A_1, C_0 and $\mathsf{first}_1^{m-1}(1, R_2 \setminus R_3) \cup \{\mathsf{last}(1, R_2)\}$ will be a monochromatic solution to $p(m, m, m; 2)$.

If $\gamma_0' \leq \beta' - \mu$, then (52) implies $\beta' \geq \frac{m-2}{3} + \mu$, contrary to (48). On the other hand, if $\gamma_0' \geq \beta' - \mu + 1$, then (52) instead yields

$$\frac{2m-4}{3} - \beta' + \mu + \nu \leq \gamma_0' \leq \gamma_0 \leq \alpha - \frac{2m-4}{3},$$

where the final inequality follows from (41) and the assumption $|\Delta^{-1}(0) \cap R_2| \leq m + \beta$. Rearranging the above inequality and applying the estimate (48), we find that $\alpha \geq m - 1 + 2\mu + 2\nu$. Since we trivially have $\beta + \alpha \leq m - 1$, we conclude that $\alpha = m - 1$ and $\beta = \mu = \nu = 0$ with equality holding in all estimates used to derive the bound $\alpha \geq m - 1 + 2\mu + 2\nu$. In particular, $0 = m - 1 - \alpha = \beta' = \frac{m-5}{3} - \mu - \nu = \frac{m-5}{3}$, contradicting that we now have $m \geq 6$. This completes Step 6.

Step 6 showed that R_2 contains enough integers colored by 0, but we now need to know something about their distribution as measured by γ_0'. With some effort, we manage this below in Step 7.

Step 7: $\gamma_0' \geq \beta + 2$.

Suppose by way of contradiction that $\gamma_0' \leq \beta + 1 \leq \beta' + 1$. Then

$$m - 1 + \beta' - \mu \leq \mathsf{diam}\, C_0 \leq m + \beta'$$

by (50). Also, in view of Step 6, there are at least m integers colored by 0 greater than y. As a result,

$$\mathsf{first}(0, R_2 \setminus R_3) \geq 5m - 3 + \lfloor \frac{2m-2}{3} \rfloor - \beta',$$

for otherwise A_1, C_0 and $\mathsf{first}_1^{m-1}(0, R_2 \setminus R_3) \cup \{6m-4+\lfloor\frac{2m-2}{3}\rfloor\}$ form a monochromatic solution to $p(m, m, m; 2)$ in view of (27) and Lemma 2.2(ii). Thus

$$\Delta([y, 5m-4+\lfloor\frac{2m-2}{3}\rfloor - \beta']) = \{1\}. \tag{53}$$

We handle two cases.

Case 7.1: $|\Delta^{-1}(1) \cap R_3| = m$.

In this case, the assumption $\gamma_0' \le \beta + 1$ yields

$$y = \min R_2 + |\Delta^{-1}(1) \cap R_3| + \gamma_0' - 1 \le 4m - 2 - \nu,$$

while (48) implies $5m - 3 + \beta' \le 5m - 4 + \lfloor\frac{2m-2}{3}\rfloor - \beta'$. Thus

$$D = [4m-2, 5m-4] \cup \{5m-3+\beta'\}$$

is a monochromatic m-subset by (53) with $\mathsf{diam}\, D = m - 1 + \beta'$. Consequently, we must now have

$$\mathsf{first}(0, R_2 \setminus R_3) \ge 5m - 2 + \lfloor\frac{2m-2}{3}\rfloor - \beta',$$

for otherwise A_1, D and $\mathsf{first}_1^{m-1}(0, R_2 \setminus R_3) \cup \{6m-4+\lfloor\frac{2m-2}{3}\rfloor\}$ form a monochromatic solution to $p(m, m, m; 2)$ in view of Step 6, (27), and Lemma 2.2(ii).

In consequence,

$$D' = [4m-1, 5m-3] \cup \{5m-3+\lfloor\frac{2m-2}{3}\rfloor - \beta'\}$$

is a monochromatic m-subset with $\min D' > y$ and $\mathsf{diam}\, D' \ge m - 2 + \frac{2m-4}{3} - \beta' > m - 1 + \beta'$ in view of (48). Thus A_1, C_0 and D' form a monochromatic solution to $p(m, m, m; 2)$ in view of Lemma 2.2(ii), completing the case.

Case 7.2: $|\Delta^{-1}(1) \cap R_3| > m$.

In this case, (49) and the assumption $\gamma_0' \le \beta + 1$ together imply that

$$\begin{aligned} y &= \min R_2 + |\Delta^{-1}(1) \cap R_3| + \gamma_0' - 1 \\ &\le 3m - 2 - \nu + |\Delta^{-1}(1) \cap R_3| \le 4m - 2 + \beta' - \gamma_0'' - \mu - \nu \end{aligned}$$

and that $B_0 = \mathsf{first}_1^m(1, R_3)$ is a monochromatic m-subset with

$$\mathsf{diam}\, B_0 \le m + \gamma_0'' - 1 \qquad \text{and} \qquad \max B_0 < y.$$

Consequently, since $5m - 3 + \beta' \le 5m - 4 + \lfloor\frac{2m-2}{3}\rfloor - \beta'$ by (48), it follows from (53) that

$$C = [4m - 2 + \beta' - \gamma_0'', 5m - 4 + \beta' - \gamma_0''] \cup \{5m - 3 + \beta'\}$$

is a monochromatic m-subset with $\min C \ge y > \max B_0$ and $\mathsf{diam}\, C = m + \gamma_0'' - 1$. Thus we must have $\mathsf{first}(0, R_2 \setminus R_3) \ge 5m-2+\lfloor\frac{2m-2}{3}\rfloor - \gamma_0''$ else B_0, C and $\mathsf{first}_1^{m-1}(0, R_2 \setminus R_3) \cup \{6m-4+\lfloor\frac{2m-2}{3}\rfloor\}$ will be a monochromatic solution to $p(m, m, m; 2)$. But that means

$$D = [4m - 1 + \beta' - \gamma_0'', 5m - 3 + \beta - \gamma_0''] \cup \{5m - 3 + \lfloor\frac{2m-2}{3}\rfloor - \gamma_0''\}$$

is a monochromatic m-subset with $\min D > y$ and $\mathsf{diam}\, D \geq m - 2 + \frac{2m-4}{3} - \beta' > m + \beta' - 1$ (in view of (48)). Hence A_1, C_0 and D form a monochromatic solution to $p(m, m, m; 2)$ in view of Lemma 2.2(ii), completing the case and Step 7.

Now it is time to finish the proof. Steps 6 and 7 allow us to define the integers z and z' below in such a way that we get a large diameter m-set involving z' without using too many elements colored by 0 beforehand and with z and z' both less than the key extremal integer y. The remaining possibilities for the coloring rapidly diminish.

Let $z = \mathsf{first}_{\beta+1}(0, R_2)$ and let $z' = \mathsf{first}_{\beta+2}(0, R_2)$. In view of Step 6, there are at least m integers colored by 0 greater than z. In view of Step 7, we have $z, z' \in R_3$ with $z < z' < y$. Thus, in view of the definition of z' and (49), we have

$$\begin{aligned} z' &\leq \min R_3 + |\Delta^{-1}(1) \cap R_3 \setminus \{y\}| + |\Delta^{-1}(0) \cap [\min R_3, z']| - 1 \\ &= \min R_3 + |\Delta^{-1}(1) \cap R_3 \setminus \{y\}| + |\Delta^{-1}(0) \cap [\min R_2, z']| - 1 \\ &= \min R_3 + |\Delta^{-1}(1) \cap R_3| + \beta \\ &\leq 4m - 2 - \nu + \beta' - \mu. \end{aligned}$$

Thus $D = \{z'\} \cup \mathsf{last}_1^{m-1}(0, R_2)$ is a monochromatic m-subset with $\min D = z' > z$ and

$$\mathsf{diam}\, D \geq 2m - 2 + \frac{2m-4}{3} - \beta' + \mu + \nu \geq 2m - 2 + \frac{m+1}{3} + 2\mu + 2\nu, \tag{54}$$

where the second inequality follows from (48).

Recall from (39) and the definition of z that $\Delta(2m - 1 - \mu - \nu) = 0$ with the interval $[2m - 1 - \mu - \nu, z]$ containing exactly m integers colored by 0 and at most

$$\mu + |\Delta^{-1}(1) \cap R_3 \setminus \{y\}| \leq \mu + m + \beta' - \mu - 1 = m + \beta' - 1$$

integers colored by 1 (in view of (49)). Thus, if there are at least $m - 1$ integers colored by 1 in $[2m - 1 - \mu - \nu, z]$, then $C = \mathsf{first}_1^m(0, [2m - 1 - \mu - \nu, z])$ will be a monochromatic m-subset with $2m - 2 \leq C \leq 2m - 2 + \beta'$, in which case A_0, C and D will form a monochromatic solution to $p(m, m, m; 2)$ in view of (54) and (48). Therefore we may instead assume there are at most $m - 2$ integers colored by 1 in $[2m - 1 - \mu - \nu, z]$.

In view of (26) and (39), there are at least $m - \alpha + \beta + \frac{2m-4}{3} + 1 + \nu \geq m - \alpha + 1$ integers colored by 1 in $[2m - 1 - \mu - \nu, z]$. Furthermore, in view of (39), we know that there are $\alpha + \beta$ integers colored by 0 less than $2m - 1 - \mu - \nu$ with at most $m - 1 - \alpha$ integers colored by 1 between $\mathsf{first}(0, R_1)$ and $2m - 1 - \mu - \nu$. Consequently, there exists a monochromatic m-subset $C' \subseteq [\mathsf{first}(0, R_1), z]$ with $2m - 2 \leq \mathsf{diam}\, C' \leq 2m - 2 + (m - 1 - \alpha) = 2m - 2 + \beta'$. Indeed, simply take C as defined in the previous paragraph and replace $\min C$ with the maximal integer colored by 0 such that the resulting m-set C' has $\mathsf{diam}\, C' \geq 2m - 2$. This integer exists as $|[\mathsf{first}(0, R_1), z]| \geq |\Delta^{-1}(0) \cap [1, z]| + |\Delta^{-1} \cap [\mathsf{first}(0, R_1), z]| \geq m + \alpha + \beta + m - \alpha + 1 \geq 2m + \beta + 1.$

But now A_0, C' and D form a monochromatic solution to $p(m, m, m; 2)$ in view of (54) and (48), completing the proof. □

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