ALGEBRAIC THEORY I HW 3: DUE FRIDAY 10/15

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Recall: $Z_0(G) = \{1\}$, $Z_1(G) = Z(G)$ and $Z_n(G) \trianglelefteq G$ is defined via the 4-th (lattice) homomorphism theorem as the unique subgroup of G containing $Z_{n-1}(G)$ such that

$$Z_n(G)/Z_{n-1}(G) = Z\big(G/Z_{n-1}(G)\big).$$

Question 1. *Let G be a group (possibly infinite) and let $\varphi : G \to \overline{G}$ be a surjective group homomorphism. Set $\overline{x} := \varphi(x)$ for the image of $x \in G$. For $X \subseteq G$, let $\overline{X} = \{\overline{x} : x \in G\}$.*

(a) *If $K \trianglelefteq G$ and $H \trianglelefteq \overline{G}$ with $\overline{K} \leq H$, show that the map $\overline{\varphi} : G/K \to \overline{G}/H$, given by $\overline{\varphi}(xK) = \overline{x}H$, is a well-defined group homomorphism.*

(b) *Show that $\overline{Z_n(G)} \leq Z_n(\overline{G})$ for all $n \geq 0$ (Hint: use induction on n).*

(c) *If G is nilpotent, show that $\overline{G}$ is also nilpotent.*

(d) *If $H \leq G$, show that $Z_n(G) \cap H \leq Z_n(H)$ for all $n \geq 0$*

(e) *If G is nilpotent and $H \leq G$, show that H is nilpotent.*

For a group G, let $\Phi(G) = \bigcap_{\substack{M<G \\ maximal}} M$ be the intersection of all maximal subgroups $M < G$. Set $\Phi(G) = G$ if G has no maximal subgroups.

Question 2. *Show that $\Phi(G)$ is a characteristic subgroup of G.*

Question 3. *Let G be a finite group.*

1. *If P is a sylow p-group in $\Phi(G)$, show that P is a normal subgroup in G.*
2. *Show that $\Phi(G)$ is nilpotent.*

For a group G, a *minimal normal subgroup* $M \leq G$ is a subgroup M which is normal in G but which contains no proper, nontrivial subgroup N which is also normal in G. In other words, $M \trianglelefteq G$, but $1 < N < M$ ensures N is not normal in G.

Question 4. *Let G be a finite solvable group. Show that every minimal normal subgroup $M \leq G$ is an elementary abelian p-group (so an abelian group all of whose non-identity elements have order p for some prime $p \geq 2$). Hint: Show that the subgroups $[M, M]$ and $\langle x^p : x \in M\rangle$ are characteristic subgroups of M.*

Question 5. *Let G be a finite solvable group and let $H < G$ be a maximal subgroup. Show that $|G : H| = p^s$ for some prime $p \geq 2$ and $s \geq 1$. Hint: Proceed by induction on $|G|$, consider a (nontrivial) minimal normal subgroup $N \leq G$, and handle two cases depending on whether $N \leq H$ or $N \not\leq H$.*