ALGEBRAIC THEORY I HW 2: DUE FRIDAY 10/01/2021

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Question 1. *Let* G_1 *and* G_2 *be finite groups with* $\gcd(|G_1|, |G_2|) = 1$*. Show that* $\mathsf{Aut}(G_1 \times G_2) \cong \mathsf{Aut}(G_1) \times \mathsf{Aut}(G_2)$*.*

Question 2. *Let* $n \geq 1$ *be an integer. For* $x \in \mathbb{Z}$*, let* $\overline{x} = x + n\mathbb{Z} \in \mathbb{Z}/n\mathbb{Z}$*, and let* $(\mathbb{Z}/n\mathbb{Z})^\times = \{\overline{x} : x \in \mathbb{Z}, \gcd(x, n) = 1\}$*.*

1. *Show that* $(\mathbb{Z}/n\mathbb{Z})^\times$ *is an abelian multiplicative group.*
2. *Show that* $\mathsf{Aut}(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$*.*

For an integer $n \geq 1$, let C_n denote a multiplicative cyclic group of order n, let S_n denote the symmetric group on n elements, and let $D_n = \langle a, b : a^n = 1,\ b^2 = 1,\ bab^{-1} = a^{-1} \rangle$ denote a dihedral group of order $2n$.

Question 3. *Let* $H = \langle x \rangle \cong C_2$ *and* $N = \langle y \rangle \cong C_{15}$ *be cyclic groups generated by* $x \in H$ *and* $y \in N$*, respectively.*

1. *Show that* $\mathsf{Aut}(C_{15}) \cong C_2 \times C_4$*.*
2. *Let* $\alpha : H \to \mathsf{Aut}(N)$ *be a homomorphism and let* $\alpha(x)(y) = y^r$*, with* $r \in \{0, 1, \ldots, 14\}$*. What possible values can* r *be?*
3. *For each possible value of* α *(and hence* r*) from Item 2, determine which of the following four groups is isomorphic to* $N \rtimes_\alpha H$*:* C_{30}*,* D_{15}*,* $C_3 \times D_5$*,* $C_5 \times S_3$*.*

Question 4. *Show there is no simple group of order* 5103*.*

Question 5. *Show there is no simple group of order* 4851*.*