ALGEBRAIC THEORY I HW 5: DUE (IN CLASS) WEDNESDAY 11/20/2019

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Question 1. *Let G be a group and let H and K be normal subgroups with $H \cap K = \{1\}$. Show that $hk = kh$ for all $h \in H$ and $k \in K$.*

Question 2. *Let G be a nontrivial group and let H be a maximal normal subgroup of G, meaning H is maximal by inclusion among all proper, normal subgroups of G. Show that G/H has no proper, nontrivial normal subgroups.*

Question 3. *Let G be a group acting transitively on the set Ω and let $\alpha : G \to \mathsf{Perm}(\Omega)$ be the corresponding homomorphism given by $\alpha(g)(x) = x^g$ for $g \in G$ and $x \in \Omega$. For any $x \in G$, show that* $\ker \alpha = \bigcap_{g \in G} gG_xg^{-1}$.

Question 4. *Let G be a group acting transitively on a finite set Ω, and let H be a normal subgroup of G. Consider the action of H on Ω inherited from G and let $\mathcal{O}_1, \ldots, \mathcal{O}_r$ be the distinct orbits of this action.*

1. *Show that there is a well-defined action of G on $\{\mathcal{O}_1, \ldots, \mathcal{O}_r\}$ defined by $\mathcal{O}_i^g = \{x^g : x \in \mathcal{O}_i\}$, with this action transitive.*
2. *Show that $|\mathcal{O}_i| = |\mathcal{O}_j|$ for all $i, j \in [1, r]$.*
3. *For $x \in \mathcal{O}_1$, show that $|\mathcal{O}_1| = |H : H \cap G_x|$ and $r = |G : HG_x|$.*

Question 5. *Let G be a group which acts transitively on a finite set Ω. Define a* block *to be a nonempty subset $B \subseteq \Omega$ such that, for every $g \in G$, B and $B^g = \{x^g : \ g \in G\}$ are either disjoint or equal.*

1. *Show that the definition $B^g = \{x^g : \ g \in G\}$, for a block B and $g \in G$, gives a well defined group action of G on the set $\Omega_B := \{B^g : \ g \in G\}$.*
2. *If B is a block with $x \in B$, then $G_x \leq G_B = \{g \in G : \ B^g = B\} \leq G$.*
3. *Show that there does not exist a block B with $1 < |B| < |\Omega|$ if and only if, for every $x \in \Omega$, the only subgroups of G containing G_x are G_x and G.*